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A METHOD FOR EVALUATING AIRCRAFT STABILITY PARAMETERS FROM FLIGHT TEST DATA

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ABSTRACT

A least-squares method for evaluating aircraft stability derivatives and transfer function constants from flight test frequency response data is derived. The normalized least-squares equations for rotating vectors are derived by minimizing the sum of the squares of the residual vectors without regard to the phase angles of these residuals.

This method was applied to longitudinal oscillation data obtained from flight tests of a USAF bomber-type B-25J aircraft. It is shown that a linear dependency in the longitudinal variables of motion must be resolved before any solution can be obtained. The assumption of a linear relationship between the stability derivatives for the forces or moments resulting from the rate of change of the angle of attack and the pitching velocity eliminated this dependency and gave acceptable results. The lift caused by pitching velocity must be included in order to obtain a reasonable value for the lift resulting from elevator deflection.

Close agreement was obtained for the related values of the transfer function constants from the data of several independent measurements.

The application of vector algebra greatly simplifies the numerical computation required to obtain the normalized equations.

PUBLICATION REVIEW

Manuscript Copy of this report has been reviewed and found satisfactory for publication.

FOR THE COMMANDING GENERAL:

E. M. Gavin

E. M. GAVIN
Colonel, USAF
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INTRODUCTION

Considerable effort has been expended in recent years to improve flight test methods of evaluating aircraft stability derivatives and transfer functions. These improved methods are required to provide more accurate data for the design of automatic control equipment and to give the aircraft designer a better check on wind tunnel and estimated parameters. Satisfactory aircraft handling qualities are also usually expressed in terms of some stability parameter, thereby necessitating flight tests to determine compliance with specifications.

Most flight-test methods of evaluating aircraft stability parameters are primarily steady-state maneuvers. A standard approach is to maintain constant as many variables of the aircraft motion as possible while measuring the effect of the remaining variables. For example, the slope of the lift curve is usually found by measuring the change in the angle of inclination of longitudinal axis with changes in the air speed in straight and level flight. This method is subject to various possible errors, however. Lift resulting from the change in elevator position and lift caused by changes in the flow field resulting from propeller "wash" and airspeed changes are included in addition to lift resulting from change in angle of attack. Also, when the lift coefficient is a function of the Mach number, this method cannot be applied. The elevator effectiveness is usually evaluated in static methods by measuring the elevator deflection required to offset a known moment applied to the aircraft. This moment change is obtained by shifting the center of gravity location. Again some error is involved in this method, since some elevator deflection is required to overcome the pitching moment resulting from changes in the aircraft angle of attack. This change in the angle of attack is a by-product of the load shift between the wing and the tail as the position of the center of gravity is changed. These methods have the further disadvantage of requiring extensive flight-test time and expert piloting technique.

The method described in this report adopts the approach of activating all of the variables of the aircraft motion simultaneously while retaining the operating conditions as constant as possible. Values of the stability parameters are then deduced in accordance with the theory of probability. It is necessary to use an equation describing the relationship between the response and the stability parameters. This relationship can be expressed in the form of equations of motion or as transfer function equations.

The flight test evaluation of the stability parameters by this method requires that each variable included in the equations be measured directly or be derived from measured data. The evaluation of the transfer function constants in the equation for the rate of pitch resulting from elevator deflection requires that the first, second, and third derivatives of the angle of pitch be obtained, as well as the elevator angle and its first derivative. With the best instrumentation techniques available to date, it has been found practicable to measure only the first pitching derivative with an acceptable degree of accuracy. The remaining derivatives must, therefore, be obtained from the measured rate by analytical, numerical, or graphical means.

Two possible techniques for exciting the variables of the aircraft motion under constant flight conditions are the transient method and the sinusoidal oscillation method. In the transient method the aircraft is disturbed from a steady-state condition by a step or pulse type motion of the aircraft control surface. The oscillation method consists of obtaining the aircraft motion by a sinusoidal variation of the control surface about a trim point at various frequencies.

The transient method requires a minimum of flight time and has the further advantage of being applicable to flight conditions other than straight and level, such as climbs, descents, or turns. However, the evaluation of the variables of motion and all required derivatives with sufficient accuracy to permit solving for the stability parameters does not appear feasible in the time plane. In the above example of the transfer function constants for the rate of pitch, $\dot{\theta}$ can be measured directly. If the time history of the rate of pitch is quite smooth, the second derivative of the pitch rate can be obtained with fair accuracy by standard numerical or graphical methods. However, if the third derivative is then obtained similarly from the second, the inaccuracies are no longer tolerable. For those elastic aircraft having bending modes that appear on the record in addition to basic airframe response, even the first derivative that is obtained by graphical or numerical methods is of doubtful value.

The oscillation method requires extensive flight-test time and is essentially restricted to straight and level-flight conditions. However, the remaining derivatives of the components of motion can be obtained analytically in the frequency plane with ease and accuracy from the phase and amplitude of any measured components. The measured component can be expressed as a rotating vector in the complex plane by the relationship $\dot{\theta} = |\dot{\theta}| e^{i\omega t}$. The second derivative is $\ddot{\theta} = \frac{d\dot{\theta}}{dt} = i\omega\dot{\theta}\, e^{i\omega t}$. Consequently, the derivative of a rotating vector can be obtained by advancing the phase angle by 90 degrees and increasing the amplitude by the factor ω. Higher order derivatives can be evaluated similarly. The oscillation method has a further advantage in that the recorded data can be corrected for the error introduced by the dynamic response characteristics of the instrumentation by applying phase and amplitude calibrations at each frequency.

The aircraft responses obtained by the two methods above can be related analytically by Fourier or Laplace Transformation theory. This provides a means of taking advantage of the best features of each technique. The flight test time can be minimized by recording measured responses in the time plane and transforming these responses to the frequency plane to take advantage of the simplified analyses procedures available. The transform theories assume that the principle of superposition applies, that is, that all components of the motion are additive and that the system can be represented by a set of linear differential equations with constant coefficients. Therefore, any nonlinearities existing in the response characteristics can, possibly, lead to prohibitive errors in the transformation from the time plane to the frequency plane. The validity of the transformation can be checked by observing the agreement in several frequency response characteristics obtained by analyzing transients having appreciable differences in frequency content. A numerical method for accurately obtaining the Fourier transforms of arbitrary time histories is given in Reference 1.

SYMBOLS AND NOTATION

δ	perturbation elevator deflection, radians
θ	perturbation angle of pitch, radians
$\dot{\theta}$	pitching velocity, $d\theta/dt$, radians/second
$\ddot{\theta}$	pitching acceleration, $d^2\theta/dt^2$, radians/second^2
α	perturbation angle of attack, radians
γ	perturbation flight path angle, radians
ω	angular frequency, radians/second
ρ	air density, slugs/feet^3
τ	$m/\rho VS$, second
ϕ	phase angle
ϵ	downwash angle at the tail, radians
$d\epsilon/d\alpha$	rate of change of downwash angle at the tail with angle of attack.
Σ	summation of scalar products
Δ	determinant of coefficients
C_L	lift coefficient, $2mg/\rho V^2 S$
C_{L_α}	rate of change of lift coefficient with angle of attack
C_{L_δ}	rate of change of lift coefficient with elevator deflection
$C_{L_{\dot{\theta}}}$	rate of change of lift coefficient with pitching velocity
$C_{L_{\dot{\alpha}}}$	rate of change of lift coefficient with rate of change of angle of attack
$C_{L_{\ddot{\alpha}}}$	rate of change of lift coefficient with angle of attack acceleration
C_m	moment coefficient, $2I_y/\rho V^2 SC$
C_{m_α}	rate of change of moment coefficient with angle of attack
C_{m_δ}	rate of change of moment coefficient with elevator deflection
$C_{m_{\dot{\theta}}}$	rate of change of moment coefficient with pitching velocity

$C_{m_{\dot{\alpha}}}$	rate of change of moment coefficient with rate of change of angle of attack
$C_{m_{\ddot{\alpha}}}$	rate of change of moment coefficient with acceleration of angle of attack
A	in-phase component (with subscripts α, δ, $\dot{\theta}$, $\dot{\alpha}$, n)
B	out-of-phase component (with subscripts α, δ, $\dot{\theta}$, $\dot{\alpha}$, n)
A, B, C, E	transfer function constants (with subscripts 0, 1, 2)
D	differential operator, d/dt
i	imaginary unit, $\sqrt{-1}$, (complex notation)
i	unit vector along vertical axis, (vector notation)
V	true velocity, feet/second
l_t	tail length, feet
I_y	aircraft moment of inertia about lateral axis, slug-feet^2
m	aircraft mass, slugs
a	aircraft vertical acceleration, feet/second^2
S	wing area, square feet
C	mean aerodynamic chord, feet
n	normal acceleration, g's
g	acceleration due to gravity, 32.2 feet/second^2
h	$2I_y/\rho V^2SC$, second^2

THEORY

Having obtained an accurate expression of the motion of all variables of interest, the problem consists of obtaining values for the associated parameters of each variable, namely; the stability derivatives or the transfer function constants. For simplicity, the theory will be developed for the stability derivatives in the lift equation assuming that lift will result only from the incremental changes in the angle-of-attack and elevator deflection. Equation 8, Appendix I, will then be written as

$$C_{L_\alpha}\alpha + C_{L_\delta}\delta = -C_L n \tag{1}$$

This equation relates the lift forces to the vertical acceleration at any instant of time, irrespective as to whether the motion is any arbitrary transient or a steady-state sinusoidal oscillation. In the latter case, the variables can be considered for any given frequency to be constant amplitude vectors rotating in the complex plane with fixed phase relationships. These amplitudes and phase angles can be experimentally determined for a number of different frequencies at a constant flight configuration.

If these data were obtained without error of any form and the above equation described the aerodynamic and inertial forces completely, it would then be possible to solve for the unknown derivatives directly. The variables could be resolved into components along the real axis for as many frequencies as there are unknowns. A simultaneous solution of these real equations would yield the accurate values of the derivatives. Since neither of these assumptions is true, this approach leads to large errors in practice. Random errors may be present that are larger than the effect of some derivatives at the particular time in the cycle selected for taking real parts. The effect of a neglected stability derivative will also be large when its associated variable has relatively large components along the real axis as compared to some included variables. Standard curve fitting theory for scalars can be extended to minimize these errors when the variables are rotating vectors.

The problem of obtaining values of the stability derivatives in the simplified lift equation given above will now be considered when the phase angles and amplitudes have been obtained from flight data for a number of frequencies. Any values that are found will not satisfy the equations exactly at each frequency point at which data are obtained. This is expressed algebraically by transposing all of the terms in equation 1 to the left-hand side and retaining the residual resulting from the inequality on the right-hand side.

$$C_{L_\alpha}\alpha_i + C_{L_\delta}\delta_i + C_L n_i = v_i; \; i = 1, 2, 3, \ldots m \tag{2}$$

where m = the total of frequency points

v = residual

Since all terms on the left are vectors, the residual, v_i, will also be a vector at each frequency. It is now required to find values of the derivatives that will make the amplitudes of these residuals small. The

relative phase angles of these residuals that result from minimizing these amplitudes are of no concern at this point. Later, it will be shown that these phase angles can be used as an indication of the validity of the equation being fitted.

It can be shown (Reference 2) that the most probable values of the unknowns will be such as to make the sum of the squares of the residuals a minimum. The square of a residual vector is a scalar equal in magnitude to the square of the amplitude of the vector.

$$\begin{aligned}\Sigma v_i^2 &= v_1^2 + v_2^2 + v_3^2 + \ldots v_m^2 \\ &= \Sigma(C_{L_\alpha}\alpha_i + C_{L_\delta}\delta_i + C_L n_i)^2 \\ &= f(C_{L_\alpha}, C_{L_\delta}) \qquad i = 1, 2, 3, \ldots m.\end{aligned} \tag{3}$$

The condition that this function be a minimum requires that its partial derivatives with respect to the unknowns be zero.

$$\frac{\partial f}{\partial C_{L_\alpha}} = 2(C_{L_\alpha}\alpha_1 + C_{L_\delta}\delta_1 + C_L n_1)\alpha_1 + 2(C_{L_\alpha}\alpha_2 + C_{L_\delta}\delta_2 + C_L n_2)\alpha_2 \tag{4}$$

$$+ \ldots 2(C_{L_\alpha}\alpha_m + C_{L_\delta}\delta_m + C_L n_m)\alpha_m = 0$$

or $$C_{L_\alpha}\Sigma\alpha_i^2 + C_{L_\delta}\Sigma\alpha_i\delta_i = -\Sigma C_L n_i \cdot \alpha_i; \; i = 1, 2, 3, \ldots m. \tag{5}$$

Similarly, when

$$\begin{aligned}&\frac{\partial f}{\partial C_{L_\delta}} = 0 \\ &C_{L_\alpha}\Sigma\alpha_i \cdot \delta_i + C_{L_\delta}\Sigma\delta_i^2 = -\Sigma C_L n_i \cdot \delta_i; \; i = 1, 2, 3, \ldots m.\end{aligned} \tag{6}$$

Equations 5 and 6 are the normalized equations for the least-squares solution for two unknown coefficients of variable vectors. The indicated products of the various pairs of vectors are defined as the scalar or dot products (Reference 3). The scalar product of two vectors is numerically equal to the product of their amplitudes times the cosine of the angle between the two vectors. This is equivalent to multiplying the vector of the variable being normalized by the respective components of the other vectors along this normalized vector. In this way each individual equation is normalized for each unknown parameter at that unique angle at which the associated vector is at its peak or maximum value. It is apparent that each parameter will have the maximum effect on the response when its vector is at its peak value. It is, therefore, logical that the parameter should be so selected that it yields a minimum error at the point where it has maximum effect. Basically, in the equations given above, this amounts to resolving the vectors into real parts with the angle-of-attack vector lying on the real axis when normalizing the equation for C_{L_α} and then resolving the vectors into real parts again with the elevator-deflection vector lying along the real axis when normalizing the equations for C_{L_δ}.

Using the definition given for the scalar products the normalized equations can be simplified.

$$\begin{aligned}\alpha \cdot \delta &= |\alpha|\,|\delta| \cos \phi_{\alpha\delta} \\ &= |\alpha|\,|\delta| \cos (\phi_{\alpha} - \phi_{\delta}) \\ &= |\alpha|\,|\delta| (\cos \phi_{\alpha} \cos \phi_{\delta} + \sin \phi_{\alpha} \sin \phi_{\delta})\end{aligned} \tag{7}$$

where $|\;|$ indicates the absolute length of the vector

$\phi_{\alpha\delta}$ is the angle between the α and δ vectors

ϕ_{α} is the angle between α and the real axis

ϕ_{δ} is the angle between δ and the real axis

Let $A_{\alpha} = |\alpha| \cos \phi_{\alpha}$, $A_{\delta} = |\delta| \cos \phi_{\delta}$,

$B_{\alpha} = |\alpha| \sin \phi_{\alpha}$, $B_{\delta} = |\delta| \sin \phi_{\delta}$.

Then $$\alpha \cdot \delta = A_{\alpha}A_{\delta} + B_{\alpha}B_{\delta} \tag{8}$$

Similarly, the remaining scalar products can be found in terms of real and imaginary components of the complex variables. Equations 5 and 6 become

$$C_{L_{\alpha}} \sum (A_{\alpha_i}^2 + B_{\alpha_i}^2) + C_{L_{\delta}} \sum (A_{\alpha_i}A_{\delta_i} + B_{\alpha_i}B_{\delta_i}) = \sum C_L (A_{\alpha_i}A_{n_i} + B_{\alpha_i}B_{n_i}) \tag{9}$$

$$C_{L_{\alpha}} \sum (A_{\alpha_i}A_{\delta_i} + B_{\alpha_i} \cdot B_{\delta_i}) + C_{L_{\delta}} \sum (A_{\delta_i}^{\,2} + B_{\delta_i}^{\,2}) = \sum C_L (A_{\delta_i}A_{n_i} + B_{\delta_i}B_{n_i}) \tag{10}$$

The amplitudes of all the vectors are usually given in terms of a unit input, and the phase angles are all referred to the input vector which, in this case, is the elevator deflection. Therefore, the elevator deflection vector will have a unit length lying along the real axis in every case. Then $A_{\delta_i} = 1$ and $B_{\delta_i} = 0$ and the normalized equations become

$$C_{L_{\alpha}} \sum (A_{\alpha_i}^2 + B_{\alpha_i}^2) + C_{L_{\delta}} \sum A_{\alpha_i} = \sum C_L (A_{\alpha_i}A_{n_i} + B_{\alpha_i}B_{n_i}) \tag{11}$$

$$C_{L_{\alpha}} \sum A_{\alpha_i} + mC_{L_{\delta}} = \sum C_L A_{n_i}; \quad i = 1, 2, 3, \ldots m. \tag{12}$$

The original vector equations can also be separated into real and imaginary parts and these scalar equations normalized by the conventional least-squares method. When all the real and imaginary equations are summed in the normalizing process, the same equations as given above will result. This approach, however, makes it difficult to visualize the basic validity and effectiveness of the method.

The least-squares method for vectors can now be applied to obtain values for all of the moment and lift derivatives included in the equations of motion in Appendix I. It is first necessary to resolve the linear dependency that exists between the variables of motion, as shown in the derivation,

before normalizing the equations. In the simplified theoretical evaluation of the stability derivatives, it is found that the $\dot{\alpha}$ and $\dot{\theta}$ derivatives are directly related by a constant equal to the rate of change in downwash angle at the tail with change in aircraft angle of attack (Reference 4). This can be expressed algebraically as

$$C_{m_{\dot{\alpha}}} = KC_{m_{\dot{\theta}}} \tag{13}$$

$$C_{L_{\dot{\alpha}}} = KC_{L_{\dot{\theta}}} \tag{14}$$

where $K = \frac{d\epsilon}{d\alpha}$, the change in downwash angle at the tail with change in aircraft angle of attack.

Substitution of equations 13 and 14 into equations 7 and 8 of Appendix I will reduce the equations to three linearly independent variables.

$$C_{m_{\alpha}}\alpha + C_{m_{\delta}}\delta + C_{m_{\dot{\theta}}} \cdot (\dot{\theta} + K\dot{\alpha}) = h\ddot{\theta} \tag{15}$$

$$C_{L_{\alpha}}\alpha + C_{L_{\delta}}\delta + C_{L_{\dot{\theta}}}(\dot{\theta} + K\dot{\alpha}) = -C_L n \tag{16}$$

The solution of these equations by least-squares will yield values of the $\dot{\theta}$ derivatives directly. These derivatives will act along the new vector, $(\dot{\theta} + K\dot{\alpha})$, and thereby include the effects of the $\dot{\alpha}$ derivatives. Since the $\dot{\alpha}$ derivatives are relatively small the effects of inaccuracies in the assumed values of K should be minor. Equations 15 and 16 will be written in matrix notation for compactness before normalizing.

$$\left[\alpha \;\; \delta \;\; (\dot{\theta} + K\dot{\alpha})\right] \begin{bmatrix} C_{j_{\alpha}} \\ C_{j_{\delta}} \\ C_{j_{\dot{\theta}}} \end{bmatrix} = -C_L n \quad \text{or} \quad h\ddot{\theta} \qquad j = m \text{ or } L \tag{17}$$

The normalized equations for m data points can be obtained directly by scalar multiplication of each equation of motion by each element in the matrix of variables in turn and summing the results.

$$\begin{bmatrix} \Sigma\alpha^2 & \Sigma\alpha\cdot\delta & \Sigma\alpha\cdot(\dot{\theta} + K\dot{\alpha}) \\ \Sigma\alpha\cdot\delta & \Sigma\delta^2 & \Sigma\delta\cdot(\dot{\theta} + K\dot{\alpha}) \\ \Sigma\alpha\cdot(\dot{\theta} + K\dot{\alpha}) & \Sigma\delta\cdot(\dot{\theta} + K\dot{\alpha}) & \Sigma(\dot{\theta} + K\dot{\alpha})^2 \end{bmatrix} \begin{bmatrix} C_{j_{\alpha}} \\ C_{j_{\delta}} \\ C_{j_{\dot{\theta}}} \end{bmatrix} \tag{18}$$

$$= \begin{bmatrix} \Sigma -C_L n\cdot\alpha \\ \Sigma -C_L n\cdot\delta \\ \Sigma -C_L n\cdot(\dot{\theta} + K\dot{\alpha}) \end{bmatrix}_{j = L} \quad \text{or} \quad \begin{bmatrix} \Sigma h\ddot{\theta}\alpha \\ \Sigma h\ddot{\theta}\delta \\ \Sigma h\ddot{\theta}\cdot(\dot{\theta} + K\dot{\alpha}) \end{bmatrix}_{j = m}$$

Expanding and simplifying the scalar products and letting $\delta = A_\delta + iB_\delta = 1$, the normalized equations become

$$\begin{bmatrix} \Sigma \alpha^2 & \Sigma A_\alpha & \Sigma A_\alpha A_{\dot\theta} + B_\alpha B_{\dot\theta} \\ \Sigma A_\alpha & m & \Sigma A_{\dot\theta} + KA_{\ddot\alpha} \\ \Sigma A_\alpha A_{\dot\theta} + B_\alpha B_{\dot\theta} & \Sigma A_{\dot\theta} + KA_{\ddot\alpha} & \Sigma (\dot\theta + K\ddot\alpha)^2 \end{bmatrix} \begin{bmatrix} C_{j_\alpha} \\ C_{j_\delta} \\ C_{j_{\dot\theta}} \end{bmatrix} \tag{19}$$

$$= \begin{bmatrix} \Sigma -C_L(A_\alpha A_n + B_\alpha B_n) \\ \Sigma -C_L A_n \\ \Sigma -C_L\ (A_n A_{\dot\theta} + B_n B_{\dot\theta}) + K(A_{\ddot\alpha} A_n + B_{\ddot\alpha} B_n) \end{bmatrix} \quad j = L$$

$$\text{or} \quad \begin{bmatrix} \Sigma -\omega h(A_\alpha B_{\dot\theta} - B_\alpha A_{\dot\theta}) \\ \Sigma -\omega h\ B_{\dot\theta} \\ \Sigma\, \omega^2 hK(A_\alpha A_{\dot\theta} + B_\alpha B_{\dot\theta}) \end{bmatrix} \quad j = m$$

Similar solutions can be made for obtaining values for the transfer function constants given in equations 12, 13 and 21, Appendix I. The normalized equations for the constants in the equation for $\dot\theta/\delta$ will be derived first. Rewriting equation 12, Appendix I, and transposing terms to eliminate fractions gives

$$A_0\dot\theta + A_1D\dot\theta + A_2D^2\dot\theta = B_0\delta + B_1D\delta. \tag{20}$$

To obtain a unique solution for the constants, it is first necessary to reduce the number of unknowns by one. Since A_2 is almost entirely made up of known quantities, it is the constant most convenient to eliminate. This can be done by dividing through by A_2. Rearranging and placing a bar over the remaining constants to indicate division by A_2, the equation becomes

$$\bar B_0\delta + \bar B_1 D\delta - \bar A_1 D\dot\theta - \bar A_0\dot\theta = D^2\dot\theta \tag{21}$$

Substituting $D = i\omega$ and writing in matrix notation gives

$$\begin{bmatrix} \delta & i\omega\delta & -i\omega\dot\theta & -\dot\theta \end{bmatrix} \begin{bmatrix} \bar B_0 \\ \bar B_1 \\ \bar A_1 \\ \bar A_0 \end{bmatrix} = \begin{bmatrix} -\omega^2\dot\theta \end{bmatrix} \tag{22}$$

The normalized equations for m data points can be written directly by the method given previously.

$$
\begin{bmatrix}
\Sigma\,\delta^2 & \Sigma\, i\omega\delta\cdot\delta & \Sigma -i\omega\mathring{\theta}\cdot\delta & \Sigma -\mathring{\theta}\cdot\delta \\
\Sigma\,\delta\cdot i\omega\delta & \Sigma (i\omega\delta)^2 & \Sigma -i\omega\mathring{\theta}\cdot i\omega\delta & \Sigma -\mathring{\theta}\cdot i\omega\delta \\
\Sigma -\delta\cdot i\omega\mathring{\theta} & \Sigma -i\omega\delta\cdot i\omega\mathring{\theta} & \Sigma (-i\omega\mathring{\theta})^2 & \Sigma -\mathring{\theta}\cdot(-i\omega\mathring{\theta}) \\
\Sigma -\delta\cdot\mathring{\theta} & \Sigma -i\omega\delta\cdot\mathring{\theta} & \Sigma -i\omega\mathring{\theta}\cdot(-\mathring{\theta}) & \Sigma (-\mathring{\theta})^2
\end{bmatrix}
\begin{bmatrix} \bar{B}_0 \\ \bar{B}_1 \\ \bar{A}_1 \\ \bar{A}_0 \end{bmatrix}
=
\begin{bmatrix}
\Sigma -\omega^2\mathring{\theta}\cdot\delta \\
\Sigma -\omega^2\mathring{\theta}\cdot i\omega\delta \\
\Sigma\,\omega^2\mathring{\theta}\cdot i\omega\mathring{\theta} \\
\Sigma\,\omega^2\mathring{\theta}\cdot\mathring{\theta}
\end{bmatrix}
\tag{23}
$$

Expanding the scalar products under the summation symbols and letting $\delta = A_\delta + iB_\delta = 1$, the normalized equations become

$$
\begin{bmatrix}
m & 0 & \Sigma\,\omega B_{\mathring{\theta}} & \Sigma -A_{\mathring{\theta}} \\
0 & \Sigma\,\omega^2 & \Sigma -\omega^2 A_{\mathring{\theta}} & \Sigma -\omega B_{\mathring{\theta}} \\
\Sigma\,\omega B_{\mathring{\theta}} & \Sigma -\omega^2 A_{\mathring{\theta}} & \Sigma\,\omega^2\mathring{\theta}^2 & 0 \\
\Sigma -A_{\mathring{\theta}} & \Sigma -\omega B_{\mathring{\theta}} & 0 & \Sigma\,\mathring{\theta}^2
\end{bmatrix}
\begin{bmatrix} \bar{B}_0 \\ \bar{B}_1 \\ \bar{A}_1 \\ \bar{A}_0 \end{bmatrix}
=
\begin{bmatrix}
\Sigma -\omega^2 A_{\mathring{\theta}} \\
\Sigma -\omega^3 B_{\mathring{\theta}} \\
0 \\
\Sigma\,\omega^2\mathring{\theta}^2
\end{bmatrix}
\tag{24}
$$

Similarly, the normalized equations for the constants in the angle of attack and normal acceleration transfer functions are found to be

$$
\begin{bmatrix}
m & 0 & \Sigma\,\omega B_\alpha & \Sigma -A_x \\
0 & \Sigma\,\omega^2 & \Sigma -\omega^2 A_\alpha & \Sigma -B_\alpha \\
\Sigma\,\omega B_\alpha & \Sigma -\omega^2 A_\alpha & \Sigma\,\omega^2\alpha^2 & 0 \\
\Sigma -A_\alpha & \Sigma -\omega B_\alpha & 0 & \Sigma\,\alpha^2
\end{bmatrix}
\begin{bmatrix} \bar{C}_0 \\ \bar{C}_1 \\ \bar{A}_1 \\ \bar{A}_0 \end{bmatrix}
=
\begin{bmatrix}
\Sigma -\omega^3 A_\alpha \\
\Sigma -\omega^3 B_\alpha \\
0 \\
\Sigma -\omega^2\alpha^2
\end{bmatrix}
\tag{25}
$$

$$
\begin{bmatrix}
m & 0 & \Sigma -A_n & \Sigma\,\omega B_n & \Sigma -\omega^2 \\
0 & \Sigma\,\omega^2 & \Sigma -\omega B_n & \Sigma -\omega^2 A_n & 0 \\
\Sigma -A_n & \Sigma -\omega^3{}_n & \Sigma\, n^2 & 0 & \Sigma\,\omega^2 A_n \\
\Sigma\,\omega B_n & \Sigma -\omega^2 A_n & 0 & \Sigma\,\omega^2 n^2 & \Sigma -\omega^3 B_n \\
\Sigma -\omega^2 & 0 & \Sigma\,\omega^2 A_n & \Sigma -\omega^3 B_n & \Sigma\,\omega^4
\end{bmatrix}
\begin{bmatrix} \bar{E}_0 \\ \bar{E}_1 \\ \bar{A}_0 \\ \bar{A}_1 \\ \bar{E}_2 \end{bmatrix}
=
\begin{bmatrix}
\Sigma -\omega^2 A_n \\
\Sigma -\omega^3 B_n \\
\Sigma\,\omega^2 n^2 \\
0 \\
\Sigma\,\omega^4 A_n
\end{bmatrix}
\tag{26}
$$

APPLICATION OF METHOD AND DISCUSSION OF RESULTS

The least-squares method for vectors was applied to actual flight test longitudinal response data to obtain values for the stability parameters. The test aircraft was a twin-engine light bomber-type B-25J and is pictured in Figure 1, Appendix I. A complete set of frequency response characteristics for the air speed, altitude, and center of gravity range of this airplane has been measured in flight. These measured responses are intended to provide a store of flight-test data for the development of data reduction methods and to investigate the possibility of extrapolating the results of limited test programs. The test methods and instrumentation are described in Reference 5.

The response data, used to demonstrate the application of the methods of this report, were obtained at an altitude of 10,000 feet, 155 mph indicated air speed, and with the center of gravity located at 27 per cent of the mean aerodynamic chord. The low air speed was selected to provide a large range of values of the reduced frequency, K, where

$$K = \frac{\omega C}{2V} \tag{27}$$

ω is the frequency of oscillation in radians per second.

C is the wing chord, feet

V is the free stream velocity, feet/second

Unsteady flow effects are, theoretically, a function of the reduced frequency. The results obtained from this example will give some indication of the possibility of evaluating these unsteady flow effects from flight-test data.

The measured frequency response characteristics of the aircraft pitching velocity and normal acceleration to sinusoidal elevator oscillations are given in Figures 2 and 3 respectively. Some of the scatter in the data is caused by the small variations in air speed, altitude, and weight that occurred during the course of the oscillation tests. The parameters and responses obtained at each frequency are listed in Table 1, Columns 1, 2, 8, 9, 15, 19, 27, and 42.

Table 1 also presents a procedure for obtaining all of the scalar products and sums of products required for the normalized least-squares equations for the lift and moment derivatives and the pitching velocity, normal acceleration, and angle-of-attack transfer function constants. When only the stability derivatives are required, this table can be reduced from 54 columns to 40.

A convenient short-cut method of solving simultaneous linear equations on a desk calculator is outlined in Table 2 and is self-explanatory. The theory and development of this method are explained in References 6 and 7. Basically, it consists of combining the steps that are normally required in reducing the equations, one variable at a time, into a complete operation for each element. The advantage, when using a desk calculator, is that the results

of intermediate steps are retained in the machine, thereby eliminating writing down the results of each step.

The solution of the normalized equations of motion for the stability derivatives is outlined in Table 3. The augmented matrix, in this case consists of the matrix of coefficients together with the right-hand columns for the lift equation and the moment equation. The matrix of the coefficients will be the same for the lift and moment equations as shown in the derivation (Appendix I). The auxiliary matrix for the coefficients needs only to be calculated once to obtain both the lift and the moment derivatives. The second-right hand column in the auxiliary matrix is found by the same steps as the first.

By arranging the variables in the equation of motion in the order of descending magnitude before normalizing, it becomes possible to check the effect of neglecting the smaller derivatives on the values obtained for the remaining derivatives. This can be done by assuming that the values of the derivatives to be neglected are zero and completing the remaining solution as before. This is shown in Table 3 where $C_{L_{\dot{\theta}}}$ is assumed equal to zero and values for the remaining lift derivatives are found. Then $C_{L_{\dot{\theta}}}$ and $C_{L_{\delta}}$ are both dropped to obtain a solution when only $C_{L_{\alpha}}$ is left. This method will not work if intermediate derivatives are assumed zero and the remaining derivatives to the right are not zero.

The values of the derivatives listed in Table 4 agree closely to values obtained by other methods where comparable data are available. The slope of the lift curve, $C_{L_{\alpha}}$, was found to be 5.111 compared to the value of 5.25 from wind tunnel results (Reference 8). The value of the pitching moment derivative, $C_{m_{\alpha}}$, equal to -.553, agrees well with the wind tunnel value of -.525 (Reference 8).

Theoretically, the lift and moment resulting from elevator deflection are related as follows (Reference 4):

$$C_{m_{\delta}} = -l_t \times C_{L_{\delta}}/C = -2.57\ C_{L_{\delta}}$$

where C = the mean aerodynamic chord = 9.69 feet for the B-25J

l_t = the effective tail length = 25.9 feet for the B-25J.

It can be seen that the results in Table 3 where $C_{m_{\delta}} = -1.418$ and $C_{L_{\delta}} = 0.556$ are in good agreement with this equation. The same relationship should hold for the pitching velocity derivatives, however, the agreement is not as good here for $C_{m_{\dot{\theta}}} = -0.270$ and $C_{L_{\dot{\theta}}} = 0.141$.

As pointed out in the previous section, the effect of the rate of change of the angle of attack is included by an assumed linear relationship with the rate of pitch derivatives. The derivative, $C_{L_{\dot{\alpha}}}$, is so small that it can be neglected without appreciable effects on the values for the remaining lift

derivatives. This was verified by actually carrying out the solution and obtaining values of 5.061, 0.526, and 0.194 for C_{L_α}, C_{L_δ}, and $C_{L_{\dot\theta}}$ respectively.

When the lift caused by pitching velocity is not included, the value found for C_{L_δ} is considerably reduced (0.287), although the relative effect on C_{L_α} is much less (5.206). The value for the slope of the lift curve when the angle of attack is the only variable included, is 5.176.

One of the reasons that the value found for C_{L_α} is only slightly affected by neglecting the smaller lift derivatives is that the sums of the products in the normalized equations are predominately composed of the responses in the low frequency range. At these frequencies, the angle of attack effect is large, whereas the amount of lift from the elevator deflection and the rate of pitch are approximately equal and 180 degrees out of phase, thereby having little net effect. At higher frequencies, these latter derivatives become increasingly important as their associated variables are relatively large and their relative phase angle changes.

At the high frequencies, the effects of other higher order derivatives such as the lift and moment resulting from the angle of attack acceleration, $C_{L_{\ddot\alpha}}$ and $C_{m_{\ddot\alpha}}$, should become appreciable. To examine these effects the normalized equations were obtained from the results of a small range of frequency and solved as in the example in Table 3. The sums of the products from six adjacent frequency points such as 1 through 6, 2 through 7, etc., were taken from Table 1. The values of the derivatives found in this way are plotted versus the median frequency of each group in Figures 4 and 5. The derivatives found in Table 3 are plotted as dashed lines over the entire frequency range to provide a comparison with the overall averages.

The angle of attack stability derivatives show consistent trends which can be attributed to the angle of attack acceleration derivatives. The variables $\ddot\alpha$ and α are vectors along the same line but 180 degrees out of phase. The response caused by the $\ddot\alpha$ stability derivatives will either increase or decrease the effective value of the α stability derivatives depending on the relative signs. The remaining parameters are surprisingly constant throughout the frequency range, considering that the theory of unsteady flow indicates variations of these derivatives with frequency. The scatter in the low frequency range is believed to be caused by errors in the data resulting from slight turbulence, although variation in the speed of the aircraft during the oscillation cycle may have contributed also.

The normalized lift equations for these limited frequency ranges were also solved neglecting first the $C_{L_{\dot\theta}}$ and $C_{L_{\dot\alpha}}$ derivatives and then also the C_{L_δ} derivative. The results are plotted in Figure 6. The importance of including these latter derivatives in the solution at high frequencies is clearly shown in this graph. Values found for the slope of the lift curve are more than double the theoretically expected size in the high frequency range when these derivatives are neglected.

The effectiveness of the least-squares process in fitting the equations of

motion to the original data can be examined by plotting the phase angles and amplitudes of the residual vectors for all frequencies. If the equations of motion include all of the forces that act on the aircraft, the residuals should consist of small vectors with random phase angles caused by the normal distribution of errors in measurement. If consistent trends do appear in the plots of the phase angles of the residuals versus frequency, it should be possible to find another stability parameter for the equation of motion. This parameter should be nearly in phase with the residuals. This effect is demonstrated in Figures 7 a and 7 b, where the phase and amplitudes of the residuals are plotted for several lift equations. The lift derivatives are obtained from Table 3.

The top graphs in Figures 7 a and 7 b are the amplitudes and phase angles of the residuals, respectively, when only the lift from angle of attack is considered. The amplitudes are quite large and the phase angles are mostly in the vicinity of 200 degrees. Now, since C_{L_α} acts along the 0 or 180 degree vector line, it would appear that a value could be obtained for this parameter. The effect of including C_{L_α} in the equation of lift is shown by the residuals in the middle graphs, Figures 7 a and 7 b. The fit has been improved, as shown by the decrease in the sum of squares of the amplitudes. However, the phase angles still show a fairly consistent trend. The inclusion of the $C_{L_{\dot\theta}}$ and $C_{L_{\dot\alpha}}$ derivatives reduces the amplitudes of the residuals still further while the scatter in the phase angles becomes considerably more random.

It is possible that further analysis will reval other derivatives that will improve the fit of the lift or moment equations to the data. Care would have to be exercised to avoid introducing linear dependencies in the variables of motion in order that valid results would be obtained.

The effects of some consistent errors in the measured flight-test data can be pointed out at this time. Calibration errors in air speed, altitude, temperature, weight, and the moment of inertia will be consistent for all frequency points. These items appear primarily on the right-hand side of the normalized equations of motion. Since the equations are linear these errors will affect all of the stability derivatives by the same percentage. The true air speed is also involved in the computation of the angle of attack, so, additional error will result if this item is not accurate. Control surface deflections are difficult to measure accurately because of twist and distortions in the surface caused by the air loads. Since only the amplitude is usually in error, the effect on the accuracy of the derivatives is minor. The surface effectiveness will be in error inversely as the error in measurement while the other parameters will not be affected.

The transfer function constants were also computed from flight test data using average values of the flight conditions and flight test values of the stability derivatives. These computations are shown in Table 7. The computed values of the constants are compared in Table 8 with those obtained directly from the normalized equations. The agreement is good for most constants although there is appreciable difference in several cases such as $\bar{B}_0$. The effect of these differences was examined by plotting the transfer functions for the pitching velocity in Figure 8. The results are in very close agreement throughout

the frequency range. The actual flight test results are plotted in Figure 8 also to show the "fit" obtained by the two methods. The transfer function equations fair the flight test data very closely in all respects except for the amplitude ratio in the low frequency range.

The individual data points could be weighted, prior to normalizing, by a factor proportional to their relative accuracy of measurement. It can be seen that this would improve the accuracy of the parameters that are obtained in the solution since the effect of the less accurate data in the normalized equations would be decreased. The errors in the data obtained from flight test are predominantly of an incremental nature resulting from reading inaccuracies. In the normalized equations for the stability derivatives, the weighting is automatically included by summing the data directly as measured.

In the normalized equations for the transfer function constants the results of high frequency oscillations comprise the largest part of the coefficients as a result of the high powers of the frequency that are involved. Since the responses in this frequency range are small these results are probably less accurate and therefore should have less weight. That this might be the case is indicated by the comparison of the plots of the transfer function equation using constants from the least-squares solution and of the actual flight data (Figure 8).

The effect of increasing the weight of low frequency data points was investigated to determine whether the "fit" in this region could be improved. This was done quite simply by including only the first 17 data points in the normalized equations. These results are also included in Table 8. In most cases these values are in close agreement with those obtained from the previous solution. The frequency response characteristics for the pitching velocity from these constants corresponded so closely to those already shown in Figure 8 that it was not possible to plot the results as a separate curve.

The consistent results obtained throughout this investigation give a strong indication of the validity and accuracy of the aerodynamic theory, flight test measurements, and data analysis methods. For example, the constants, $\bar{A}_0$ and $\bar{A}_1$, are common to each of the three transfer function equations investigated. The data for the rate of pitch and the normal acceleration were obtained simultaneously but by independent measurements. Yet the agreement in the values of these constants obtained from these two sources is quite good as also is the value found by substituting stability derivatives and flight conditions into the theoretical expression for these constants (Table 8).

The plots of the derivatives in Figures 4 and 5 are other illustrations of these consistent results. Since the data selected for each set of derivatives covered only a very small frequency range the equations for the six data points in each group are nearly the same in magnitude. In other words, the equations are nearly parallel and therefore the results are highly sensitive to errors. The values plotted in these figures show little scatter, however. The magnitude of the normal acceleration residuals in the lower graph, Figure 7 a, compared to the size of the measured data in Figure 3 is another example. Note that the scale in Figure 7 a is five times as large as Figure 3.

The method of least squares for vectors also provides a possibility of solving for the aircraft longitudinal moment of inertia. The transfer function A_2 is directly proportional to the moment of inertia. All the remaining constants are divided by A_2 in the solution of the normalized equations. Therefore, the values found for those constants which are not a function of I_y before division by A_2 will be inversely proportional to the moment of inertia. These include $\overline{A}_0$, $\overline{B}_0$, $\overline{B}_1$, $\overline{C}_0$ $\overline{E}_0$, and $\overline{E}_1$. By keeping all the other flight conditions constant and obtaining data at two values of the moment of inertia differing by a known increment, it is possible to arrive at an explicit solution for the total moment of inertia for each of these constants.

CONCLUSIONS

The method of least-squares for vectors provides a method for evaluating aircraft longitudinal stability derivatives from frequency response data provided the linear dependency existing between the variables of motion is eliminated. The assumption of a linear relationship between the angle of attack rate and pitching rate lift and moment derivatives eliminated this dependency and gave acceptable results. This method of least-squares can also be used to find values from flight test frequency response data for the constants in the aircraft transfer function equations.

The lift caused by pitching velocity must be included for conventional aircraft in order to obtain reasonable values for the lift resulting from elevator deflection.

The use of vector algebra greatly simplifies the numerical computation required to obtain the normalized equations.

APPENDIX I

DERIVATION OF LONGITUDINAL DYNAMIC RESPONSE EQUATIONS

The motion of an aircraft in flight is prescribed by aerodynamic, propulsive, gravitational, and inertial forces. The equations of motion of an aircraft for small perturbations from a steady-state flight condition have been derived in considerable detail in Reference 9. For the dynamic motion considered in this report, the incremental changes in forward velocity are considered negligible. The system of axes are fixed in the body during the motion with the longitudinal or X axis being parallel to the flight path during level flight. Nose-up moments and angular displacements, velocities, and accelerations are positive. Normal acceleration and force components are positive downwards. Elevator displacements and hinge moments are positive when the trailing edge is moved downwards.

The equations for the pitching and normal acceleration degrees of freedom can now be derived in the form that was found most convenient for analyzing flight-test response data.

$$M = I_y \ddot{\theta} \qquad (1)$$

$$F_z = m a_z \qquad (2)$$

where M is the sum of the aerodynamic moments about the lateral axis, pound feet.

I_y is the aircraft moment of inertia about the lateral axis, slug-feet^2

$\ddot{\theta}$ is the aircraft pitching acceleration, radian/second^2

F_z is the sum of the aerodynamic forces along the vertical axis, pounds

m is the aircraft mass, slugs

a_z is the aircraft vertical acceleration, feet/second^2

The important incremental variables of aircraft motion that are capable of producing aerodynamic moments about the lateral axis are the angle of attack (α), the elevator deflection (δ), the pitching velocity ($\dot{\theta}$), and the rate of change of angle of attack ($\dot{\alpha}$). Usually, only the angle-of-attack and the elevator deflection are considered important in producing normal accelerations. In the higher frequencies of the range being considered, the lift resulting from pitching velocity also becomes an appreciable part of the total lift and is, therefore, included in the development of the equations of motion. The lift resulting from $\dot{\alpha}$ has small effect on the aircraft motion but, since its inclusion simplifies the analysis of the flight-test data, it will be retained here.

Equations 1 and 2 can now be written in the form of moment and lift coefficient equations.

$$C_{m_\alpha}\frac{1}{2}\rho V^2SC\alpha + C_{m_\delta}\frac{1}{2}\rho V^2SC\delta + C_{m_{\dot\theta}}\frac{1}{2}\rho V^2SC\dot\theta + C_{m_{\dot\alpha}}\frac{1}{2}\rho V^2SC\dot\alpha = I_y\ddot\theta \tag{3}$$

$$C_{L_\alpha}\frac{1}{2}\rho V^2S\alpha + C_{L_\delta}\frac{1}{2}\rho V^2S\delta + C_{L_{\dot\theta}}\frac{1}{2}\rho V^2S\dot\theta + C_{L_{\dot\alpha}}\frac{1}{2}\rho V^2S\dot\alpha = -mgn \tag{4}$$

where C_{m_i} or C_{L_i}, $i = \alpha, \delta, \dot\theta,$ or $\dot\alpha$ are the respective moment or lift coefficients or, more generally, stability derivatives.

ρ is the air density, slugs/feet^3

V is the true velocity, feet/second

S is the wing area, square feet

C is the mean aerodynamic chord (mac), feet

n is the normal acceleration, g's

g is the acceleration due to gravity, 32.2 feet/second^2

Transposing the common terms to the right hand side and letting

$$h = 2I_y/\rho V^2SC, \text{ second}^2 \tag{5}$$

$$C_L = 2mg/\rho V^2S \tag{6}$$

the equations become

$$C_{m_\alpha}\alpha + C_{m_\delta}\delta + C_{m_{\dot\theta}}\dot\theta + C_{m_{\dot\alpha}}\dot\alpha = h\ddot\theta \tag{7}$$

$$C_{L_\alpha}\alpha + C_{L_\delta}\delta + C_{L_{\dot\theta}}\dot\theta + C_{L_{\dot\alpha}}\dot\alpha = -C_L n \tag{8}$$

If the elevator deflection δ, is considered as an input forcing function, a simultaneous algebraic solution of these equations of motion will define the response of any variable of the aircraft motion. This solution will be expressed as the ratio of two polynomials in D, the differential operator, d/dt, and corresponds to the transfer function equation as given in servomechanism theory.

The normal acceleration, a_z, is related to the rate of change in the angle between the horizontal and the tangent to the flight path, γ. This is a result of the fact that velocity is a vector and an acceleration occurs when the direction of the velocity is changed.

$$a_z = ng = V\dot\gamma = V(\dot\alpha - \dot\theta) \text{ or } n = \frac{V}{g}(\dot\alpha - \dot\theta) \tag{9}$$

The equations can now be expressed in operational form.

$$\begin{aligned} &(C_{m_\alpha} + C_{m_{\dot\alpha}}D)\ \alpha + (C_{m_{\dot\theta}} - hD)\dot\theta = -C_{m_\delta}\delta \\ &(C_{L_\alpha} + C_{L_{\dot\alpha}}D + 2\tau D)\ \alpha + (C_{L_{\dot\theta}} - 2\tau)\dot\theta = -C_{L_\delta}\delta \end{aligned} \tag{10}$$

where $\tau = m/\rho VS$, second.

Solution of the operational equations by use of determinants yields the transfer functions

$$\frac{\dot{\theta}}{\delta} = \frac{B_0 + B_1 D}{A_0 + A_1 D + A_2 D^2} \tag{12}$$

$$\frac{\alpha}{\delta} = \frac{C_0 + C_1 D}{A_0 + A_1 D + A_2 D^2} \tag{13}$$

where
$$A_0 = C_{L_\alpha} C_{m_{\dot{\theta}}} - C_{m_\alpha} C_{L_{\dot{\theta}}} + 2\tau C_{m_\alpha} \tag{14}$$

$$A_1 = C_{L_{\dot{\alpha}}} C_{m_{\dot{\theta}}} + 2\tau C_{m_{\dot{\theta}}} - hC_{L_\alpha} - C_{m_{\dot{\alpha}}} C_{L_{\dot{\theta}}} + 2\tau C_{m_{\dot{\alpha}}} \tag{15}$$

$$A_2 = -hC_{L_{\dot{\alpha}}} - 2\tau h \tag{16}$$

$$B_0 = C_{L_\delta} C_{m_\alpha} - C_{L_\alpha} C_{m_\delta} \tag{17}$$

$$B_1 = C_{m_{\dot{\alpha}}} C_{L_\delta} - C_{L_{\dot{\alpha}}} C_{m_\delta} - 2\tau C_{m_\delta} \tag{18}$$

$$C_0 = C_{m_\delta} C_{L_{\dot{\theta}}} - C_{L_\delta} C_{m_{\dot{\theta}}} - 2\tau C_{m_\delta} \tag{19}$$

$$C_1 = hC_{L_\delta} \tag{20}$$

The transfer function for the normal acceleration is obtained by combining equations 12 and 13.

$$\frac{n}{\delta} = \frac{V}{g}\left(\frac{D\alpha}{\delta} - \frac{\dot{\theta}}{\delta}\right) = \frac{E_0 + E_1 D + E_2 D^2}{A_0 + A_1 D + A_2 D^2} \tag{21}$$

where
$$E_0 = -VB_0/g = V(C_{m_\delta} C_{L_\alpha} - C_{L_\delta} C_{m_\alpha})/g \tag{22}$$

$$E_1 = V(C_0 - B_1)/g = V(C_{m_\delta} C_{L_{\dot{\theta}}} - C_{m_{\dot{\alpha}}} C_{L_\delta} - C_{L_\delta} C_{m_{\dot{\theta}}} + C_{L_{\dot{\alpha}}} C_{m_\delta})/g \tag{23}$$

$$E_2 = VC_1/g = VhC_{L_\delta}/g \tag{24}$$

The amplitude ratio and phase angle lag of any variable in response to a sinusoidal oscillation of the elevator can be obtained from the transfer function equations. These will be derived for the pitching rate only, as the procedure is the same for the remaining variables. The elevator angle is expressed as a function of time

$$\delta = |\delta| \sin \omega t \tag{25}$$

where $|\delta|$ is the peak positive value of the elevator deflection in radians.

ω is the angular velocity of the oscillation in radians per second.

$$\text{Then } \dot{\theta} = \frac{B_0 + B_1 D}{A_0 + A_1 D + A_2 D^2} \; |\delta| \; \sin \omega t \tag{26}$$

For sinusoidal motion the differential operator, D can be replaced by $i\omega$, where i is the imaginary unit, $\sqrt{-1}$.

$$\text{Now } \dot{\theta} = \frac{B_0 + i\omega B_1}{(A_0 - A_2\omega^2) + i\omega A_1} \; |\delta| \; \sin \omega t \tag{27}$$

The complex numbers in the numerator and denominator can be expressed as vectors in the complex plane having an amplitude and a phase angle, that is,

$$A + iB = r e^{i\phi}$$

$$\text{where } r = \sqrt{A^2 + B^2} \tag{28}$$

$$\phi = \tan^{-1} \frac{B}{A}$$

$$\text{Therefore } \dot{\theta} = \frac{\sqrt{B_0^2 + \omega^2 B_1^2} \cdot e^{i\phi_1}}{\sqrt{(A_0 - A_2\omega^2)^2 + \omega^2 A_1^2} \; e^{i\phi_2}} \; |\delta| \sin \omega t$$

$$= \sqrt{\frac{B_0^2 + \omega^2 B_1^2}{(A_0 - A_2\omega^2)^2 + \omega^2 A_1^2}} \; |\delta| \; \sin (\omega t - \phi_1 + \phi_2) \tag{29}$$

$$\text{where } \phi_1 = \tan^{-1} \frac{-\omega B_1}{B_0}$$

$$\phi_2 = \tan^{-1} \frac{-\omega A_1}{A_0 - A_2\omega^2}$$

The condition that must be satisfied in order that a unique set of values will exist for the stability derivatives in equations 7 and 8 is that the coefficients of the derivatives must be linearly independent. This is expressed mathematically by stating that the determinant of the coefficients of n equations with n unknowns shall be non-zero. The determinant of coefficients that must be considered in this case is the same for the moment and lift equations and is

$$\begin{vmatrix} \alpha_1 & \delta_1 & \dot{\theta}_1 & \dot{\alpha}_1 \\ \alpha_2 & \delta_2 & \dot{\theta}_2 & \dot{\alpha}_2 \\ \alpha_3 & \delta_3 & \dot{\theta}_3 & \dot{\alpha}_3 \\ \alpha_4 & \delta_4 & \dot{\theta}_4 & \dot{\alpha}_4 \end{vmatrix} = \Delta \tag{30}$$

This determinant can be shown to be equal to zero because of the relationship

given by equations 8 and 9. Solving equation 8 for α gives

$$\alpha_i = -\frac{C_L}{C_{L_\alpha}} n_i - \frac{C_{L_\delta}}{C_{L_\alpha}} \delta_i - \frac{C_{L_{\dot\theta}}}{C_{L_\alpha}} \dot\theta_i - \frac{C_{L_{\dot\alpha}}}{C_{L_\alpha}} \dot\alpha \tag{31}$$

$$= K_1 n_i + K_2 \delta_i + K_3 \dot\theta_i + K_4 \dot\alpha_i$$

where $K_1 = -\frac{C_L}{C_{L_\alpha}}$; etc. $i = 1, 2, 3, 4$.

Substitution of equation 31 in Δ gives

$$\Delta = \begin{vmatrix} (K_1 n_1) \delta_1 \dot\theta_1 \dot\alpha_1 \\ (K_1 n_2) \delta_2 \dot\theta_2 \dot\alpha_2 \\ (K_1 n_3) \delta_3 \dot\theta_3 \dot\alpha_3 \\ (K_4 n_4) \delta_4 \dot\theta_4 \dot\alpha_4 \end{vmatrix} + \begin{vmatrix} (K_2 \delta_1) \delta_1 \dot\theta_1 \dot\alpha_1 \\ (K_2 \delta_2) \delta_2 \dot\theta_2 \dot\alpha_2 \\ (K_2 \delta_3) \delta_3 \dot\theta_3 \dot\alpha_3 \\ (K_2 \delta_4) \delta_4 \dot\theta_4 \dot\alpha_4 \end{vmatrix} + \begin{vmatrix} (K_3 \dot\theta_1) \delta_1 \dot\theta_1 \dot\alpha_1 \\ (K_3 \dot\theta_2) \delta_2 \dot\theta_2 \dot\alpha_2 \\ (K_3 \dot\theta_3) \delta_3 \dot\theta_3 \dot\alpha_3 \\ (K_3 \dot\theta_4) \delta_4 \dot\theta_4 \dot\alpha_4 \end{vmatrix} + \begin{vmatrix} (K_4 \dot\alpha_1) \delta_1 \dot\theta_1 \dot\alpha_1 \\ (K_4 \dot\alpha_2) \delta_2 \dot\theta_2 \dot\alpha_2 \\ (K_4 \dot\alpha_3) \delta_3 \dot\theta_3 \dot\alpha_3 \\ (K_4 \dot\alpha_4) \delta_4 \dot\theta_4 \dot\alpha_4 \end{vmatrix} \tag{32}$$

The three determinants to the right are equal to zero since each has two columns that are in constant proportion. In the same manner the remaining determinant can also be shown to be equal to zero by the substitution of $\frac{V}{g}(\dot\alpha_i - \dot\theta_i)$ for n_i in equation 32.

TABLE 1 CALCULATION AND SUMMATION OF SCALAR PRODUCTS FOR THE NORMALIZED EQUATIONS

	1	2	3	4	5	6	7	8	9
	$\lvert n\rvert / \lvert\delta\rvert$	$\phi_{n\delta}$	Cos $\phi_{n\delta}$	Sin $\phi_{n\delta}$	$\lvert n\rvert^2$	A_n	B_n	$\lvert\dot\theta\rvert / \lvert\delta\rvert$	$\phi_{\dot\theta\delta}$
Pt No					$(1)^2$	$(1) \times -(3)$	$(1) \times (4)$		
1	12.802	-24.7	.90851	-.41787	163.89120	11.63072	-5.34953	1.978	-170.2
2	12.866	-36.7	.80178	-.59763	165.53396	10.31565	-7.68904	2.434	-171.5
3	11.517	-51.8	.61841	-.78586	132.64129	7.12221	-9.05072	2.580	-174.6
4	11.309	-65.8	.40992	-.91212	127.89348	4.63582	-10.31517	2.859	-185.0
5	10.227	-76.8	.22835	-.97358	104.59153	2.33535	-9.95679	2.823	-193.7
6	9.325	-86.8	.05582	-.99844	86.95563	.52054	-9.31046	2.850	-199.8
7	8.716	-96.0	-.10453	-.99452	75.96866	-.91107	-8.66825	2.871	-205.9
8	7.759	-102.4	-.21474	-.97667	60.20208	-1.66613	-7.57710	2.707	-210.9
9	6.861	-113.6	-.40035	-.91636	47.07332	-2.74679	-6.28717	2.543	-219.1
10	5.228	-121.2	-.51803	-.85536	27.33198	-2.70825	-4.47184	2.291	-223.9
11	4.853	-125.3	-.57786	-.81614	23.55161	-2.80435	-3.96072	2.125	-229.3
12	4.084	-133.7	-.69008	-.72297	16.67906	-2.81830	-2.95260	1.922	-234.4
13	3.065	-138.6	-.75011	-.66131	9.39423	-2.29909	-2.02692	1.702	-237.9
14	2.609	-142.3	-.79122	-.61153	6.80688	-2.06430	-1.59547	1.523	-241.4
15	2.261	-148.9	-.85627	-.51653	5.11212	-1.93602	-1.16788	1.365	-245.6
16	1.974	-152.2	-.88458	-.46639	3.89668	-1.74616	-.92065	1.208	-247.0
17	1.742	-154.4	-.90183	-.43209	3.03456	-1.57099	-.75269	1.102	-253.4
18	1.620	-155.7	-.91140	-.41151	2.62440	-1.47647	-.66665	.924	-251.8
19	1.420	-155.7	-.91140	-.41151	2.01640	-1.29419	-.58435	.887	-254.8
20	1.399	-157.0	-.92051	-.39073	1.95720	-1.28779	-.54663	.799	-255.3
21	1.262	-161.6	-.94888	-.31565	1.59264	-1.19748	-.39835	.764	-258.3
22	1.194	-159.5	-.93667	-.35021	1.42564	-1.11839	-.41815	.760	-260.0
					1070.17454	6.91452			

TABLE 1 (Contd)

Pt No	10	11	12	13	14	15	16	17
	Cos $\phi_{\dot{\theta}\delta}$	Sin $\phi_{\dot{\theta}\delta}$	$\lvert\dot{\theta}\rvert^2$	$A_{\dot{\theta}}$	$B_{\dot{\theta}}$	V	g/V	$A_{\dot{\alpha}}$
			(8)^2	(8) x (10)	(8) x (11)		32.2/(15)	(16)(6) + (13)
1	-.98541	-.17021	3.91248	-1.94914	-.33667	269.0	.11970	-.55696
2	-.98902	-.14781	5.92436	-2.40727	-.35977	266.8	.12069	-1.16227
3	-.99556	-.09411	6.65640	-2.56855	-.24280	265.3	.12135	-1.70424
4	-.99620	.08716	8.17388	-2.84812	.24918	267.5	.12036	-2.29017
5	-.97155	.23684	7.96933	-2.74268	.66859	266.8	.12069	-2.46084
6	-.94088	.33874	8.12250	-2.68151	.96540	266.8	.12069	-2.61869
7	-.89956	.43680	7.73396	-2.50167	1.21475	263.9	.12203	-2.61285
8	-.86690	.49849	7.32785	-2.34669	1.34941	267.5	.12036	-2.54722
9	-.77605	.63068	6.46685	-1.97349	1.60381	264.6	.12169	-2.30775
10	-.72055	.69340	5.24868	-1.65078	1.58858	264.6	.12169	-1.98036
11	-.65210	.75813	4.51563	-1.38571	1.61104	269.0	.11970	-1.72138
12	-.58212	.81310	3.69408	-1.11884	1.56278	266.1	.12102	-1.45991
13	-.53140	.84712	2.89680	-.90444	1.44180	266.8	.12069	-1.18192
14	-.47869	.87798	2.31953	-.72905	1.33717	265.3	.12135	-.97956
15	-.41310	.91068	1.86323	-.56388	1.24308	266.8	.12069	-.79755
16	-.39073	.92051	1.45926	-.47200	1.11197	265.3	.12135	-.68391
17	-.28569	.95832	1.21440	-.31483	1.05607	266.8	.12069	-.50443
18	-.31234	.94997	.85378	-.28860	.87777	262.4	.12271	-.46978
19	-.26219	.96502	.78677	-.23256	.85597	266.8	.12069	-.38876
20	-.25376	.96727	.63840	-.20275	.77285	266.1	.12102	-.35860
21	-.20279	.97922	.58370	-.15493	.74813	262.4	.12271	-.30188
22	-.17365	.98481	.44890	-.11634	.65982	263.1	.12237	-.25320
			88.81077	-30.15384				

TABLE 1 (Contd)

	18	19	20	21	22	23	24	25
	$B_{\dot{\alpha}}$	ω	B_{α}	A_{α}	$\lvert\alpha\rvert^2$	ω^2	ω^4	$\omega B_{\dot{\theta}}$
Pt No	(16)(7) + (14)		-(17) / (19)	(18) / (19)	$(20)^2 + (21)^2$	$(19)^2$	$(23)^2$	(19)(14)
1	-.97700	.6413	.86844	-1.52337	3.07485	.4113	.169	-.21592
2	-1.28776	.9412	1.23495	-1.36828	3.39729	.8858	.785	-.33860
3	-1.34114	1.2356	1.37934	-1.08546	3.08080	1.5266	2.330	-.29999
4	-.99231	1.5112	1.51550	-.65665	2.02792	2.2836	5.215	.37655
5	-.53304	1.7825	1.38054	-.29904	1.99532	3.1774	10.096	1.19178
6	-.15828	2.0176	1.29790	-.07845	1.69069	4.0709	16.572	1.94784
7	.15696	2.2218	1.17602	.07065	1.38800	4.9363	24.367	2.69890
8	.43735	2.4884	1.02364	.17576	1.07873	6.1921	38.342	3.35786
9	.83871	2.7834	.82910	.30132	.77820	7.7475	60.024	4.46411
10	1.04439	3.1078	.63721	.33605	.51897	9.6587	93.291	4.93708
11	1.13695	3.4088	.52499	.33354	.36626	11.6196	35.016	5.49163
12	1.20545	3.9373	.37079	.30616	.23122	15.5026	240.332	6.15320
13	1.19717	4.4018	.26851	.27198	.14607	19.3755	375.409	6.34646
14	1.14355	5.0169	.19525	.22794	.09008	25.1696	633.508	6.70848
15	1.10213	5.6045	.14231	.19665	.05892	31.4104	986.611	6.96686
16	1.00025	6.2947	.10865	.15890	.03706	39.6227	1569.962	6.99947
17	.94109	6.9765	.07230	.13489	.02342	48.6716	2368.920	7.36769
18	.79597	8.1140	.05790	.09810	.01298	65.8362	4334.403	7.12221
19	.78544	8.5418	.04551	.09195	.01053	72.9629	5323.579	7.31154
20	.70669	9.3459	.03833	.07554	.00718	87.5136	7658.629	7.22988
21	.69924	10.0339	.03009	.06969	.00576	100.6781	10136.089	7.50658
22	.60865	10.9462	.02313	.05560	.00363	119.8182	14356.401	7.22250
				-2.10654	20.72385	679.0712	48370.050	100.54611

TABLE 1 (Contd)

Pt No	26	27	28	29	30	31	32	33
	ωB_n	h	ωh	$\omega^2 \dot{\theta}^2$	$\omega^2 A_{\dot{\theta}}$	$\omega^2 A_\alpha$	$\omega^2 \alpha^2$	$\omega^2 A_n$
	19 7		19 27	23 12	23 13	23 21	23 22	23 6
1	-3.43087	.1758	.11275	1.60927	-.80171	-.62659	1.26474	4.78391
2	-7.23654	.1806	.16997	5.24758	-2.13227	-1.21197	3.00919	9.13722
3	-11.18261	.1816	.22438	10.16155	-3.92111	-1.65704	4.70310	10.87284
4	-15.58797	.1793	.27095	18.66616	-6.50407	-1.49955	6.22958	10.58652
5	-17.74818	.1806	.32192	25.32157	-8.71454	-.95016	6.33988	7.42028
6	-18.78516	.1806	.36439	33.06565	-10.91609	-.31935	6.88257	2.11905
7	-19.25895	.1829	.40636	38.17720	-12.34901	.34873	6.85161	-4.49730
8	-18.85706	.1793	.44617	45.37488	-14.53097	1.08830	6.67960	-10.31686
9	-17.49995	.1829	.50909	50.10216	-15.28965	2.33449	6.02914	-21.28089
10	-13.89781	.1840	.57184	50.69554	-15.94444	3.24581	5.01256	-26.15818
11	-13.50114	.1781	.60710	52.46996	-16.10144	3.87557	4.25522	-32.58549
12	-11.62538	.1814	.72014	57.26808	-17.34498	4.74628	3.58446	-43.69103
13	-8.92201	.1806	.79496	56.12693	-17.52397	5.26966	2.83016	-44.54594
14	-8.00438	.1829	.91760	58.38159	-18.34984	5.73711	2.26725	-51.95765
15	-6.54538	.1806	1.01217	58.52458	-17.71190	6.17688	1.85076	-60.81109
16	-5.79517	.1829	1.15129	57.82005	-18.70206	6.29621	1.46822	-69.18777
17	-5.25117	.1806	1.25996	59.10693	-15.32317	6.56550	1.14008	-76.46267
18	-5.40919	.1865	1.51325	56.20936	-19.00019	6.45840	.85422	-97.20535
19	-4.99142	.1806	1.54265	57.40492	-16.96839	6.70915	.76808	-94.42795
20	-5.11368	.1816	1.69884	55.86877	-17.74364	6.61104	.62800	-112.69878
21	-3.99697	.1853	1.85927	58.76543	-15.59796	7.01606	.58011	-120.56017
22	-4.57710	.1855	2.02832	53.78639	-13.94013	6.66237	.43458	-134.00300
	227.21809			960.15451	-295.41151	72.87690	73.66311	-955.47050

TABLE 1 (Contd)

Pt No	34	35	36	37	38	39	40
	$\omega^2 n^2$	$\omega^3 B_{\dot\theta}$	$\omega^3 B_\alpha$	$\omega^3 B_n$	$\omega^4 A_n$	$A_\alpha A_{\dot\theta} + B_\alpha B_{\dot\theta}$	$A_\alpha A_n + B_\alpha B_n$
	23 5	23 25	- 23 17	23 26	23 33	21 13 + 20 14	21 6 + 20 7
1	67.4112	-.0888	.22909	-1.4112	1.968	2.67689	-22.36366
2	146.6239	-.2999	1.02950	-6.4099	8.093	2.84952	-23.61026
3	202.4881	-.4580	2.60167	-17.0712	16.598	2.45315	-20.21487
4	292.0620	.8599	5.22992	-35.5972	24.176	2.24786	-18.67672
5	332.3268	3.7867	7.81902	-56.3927	23.577	1.74319	-14.44412
6	353.9851	7.9294	10.66034	-76.4720	8.626	1.46335	-12.12485
7	375.0045	13.3226	12.89782	-95.0681	-22.200	1.25183	-10.25837
8	372.7782	20.7922	15.77267	-116.7651	-63.883	.96886	-8.04997
9	364.7023	34.5858	17.87938	-135.5815	-164.875	.73507	-6.04035
10	263.9920	47.6858	19.12769	-134.2350	-252.654	.45752	-3.75961
11	273.6613	63.8108	20.00186	-156.8784	-378.632	.35137	-2.93547
12	258.5695	95.3908	22.63254	-180.2241	-677.327	.23692	-1.95763
13	182.0175	122.9656	22.90025	-172.8682	-863.098	.14115	-1.16955
14	171.3264	168.8496	24.65514	-201.4670	-1307.753	.09491	-.78205
15	160.5736	218.8315	25.05126	-205.5929	-1910.099	.06601	-.54692
16	154.3970	277.3383	27.09833	-229.6204	-2741.409	.04581	-.37750
17	147.6969	358.5967	24.55134	-255.5826	-3721.557	.03389	-.26634
18	172.7805	468.8994	30.92809	-356.1204	-6399.629	.02251	-.18344
19	147.1223	533.4710	28.36473	-364.1882	-6889.733	.01757	-.14560
20	171.2817	632.7131	31.38239	-447.5166	-9862.675	.01431	-.11824
21	160.3444	755.7490	30.39256	-402.4079	-12137.774	.01171	-.09544
22	170.8171	865.3870	30.33873	-548.4199	-16055.998	.00879	-.07186
	4941.9625	4690.1187	411.54432	-4195.8905	-63366.258	17.89218	

TABLE 1 (Contd)

Pt No	41	42	43	44	45	46	47	48
	$A_n \dot{A}_\theta + B_n \dot{B}_\theta$		$C_{L\alpha} \cdot n$	$C_L \dot{\theta} \cdot n$	$C_L\ A_n$	$A_\alpha \dot{B}_\theta - B_\alpha \dot{A}_\theta$	$h \dot{\theta} \alpha$	$\omega h\ \dot{B}_\theta$
	(6)(13) + (7)(14)	C_L	(42)(40)	(42)(41)	(42)(6)	(21)(11) - (20)(13)	(28)(46)	(28)(14)
1	-20.86882	.7118	-15.91846	-14.85443	8.27875	2.20558	.24868	-.03796
2	-22.06624	.7199	-16.99703	-15.88549	7.42624	3.46511	.58897	-.06115
3	-16.09624	.7348	-14.85388	-11.82751	5.23340	3.80645	.85408	-.05448
4	-15.77370	.7151	-13.35572	-11.27977	3.31507	4.15270	1.12519	.06752
5	-13.06217	.7303	-10.54854	-9.53930	1.70550	3.58645	1.15456	.21524
6	-10.38418	.7213	-8.74565	-7.49011	.37547	3.40459	1.24059	.35178
7	-8.25054	.7390	-7.58094	-6.09715	-.67328	3.02782	1.23040	.49363
8	-6.31592	.7164	-5.76700	-4.52472	-1.19362	2.63933	1.17759	.60206
9	-4.66266	.7376	-4.45506	-3.43894	-2.02590	2.11948	1.07901	.81649
10	-2.63318	.7355	-2.76520	-1.93670	-1.99191	1.58574	.90680	.90842
11	-2.49485	.7169	-2.10444	-1.78856	-2.01044	1.23710	.75105	.97806
12	-1.46104	.7335	-1.43592	-1.07167	-2.06722	.89331	.64331	1.12542
13	-.84303	.7267	-.84991	-.61263	-1.67075	.63499	.50479	1.14617
14	-.62844	.7337	-.57379	-.46109	-1.51458	.44714	.41029	1.22698
15	-.36008	.7263	-.39723	-.26153	-1.40613	.32470	.32865	1.25822
16	-.19954	.7340	-.27709	-.14646	-1.28168	.22798	.26247	1.28020
17	-.30031	.7258	-.19331	-.21796	-1.14021	.16522	.20817	1.33060
18	-.15906	.7483	-.13727	-.11903	-1.10485	.10282	.15559	1.32829
19	-.19921	.7256	-.10565	-.14454	-.93907	.08929	.13775	1.32046
20	-.16136	.7291	-.08621	-.11765	-.93893	.06616	.11239	1.31295
21	-.11249	.7445	-.07105	-.08375	-.89153	.05680	.10560	1.39097
22	-.14579	.7444	-.05349	-.10852	-.83253	.03938	.07988	1.33833
			-107.27281	-92.00751	4.65182		13.30576	18.33820

TABLE 1 (Contd)

Pt No	49	50	51	52	53	54	55
	$A_{\dot\theta} + .45A_{\dot\alpha}$	$B_{\dot\theta} + .45B_{\dot\alpha}$	$(\dot\theta + .45\dot\alpha)^2$	$n\cdot(\dot\theta + .45\dot\alpha)$	$C_L n\cdot(\dot\theta + .45\dot\alpha)$	$.45A_{\dot\theta}B_{\dot\alpha} - .45A_{\dot\alpha}B_{\dot\theta}$	$.45\omega^2 h\alpha\cdot\dot\theta$
	(13) + .45 (17)	(14) + .45 (18)	(49)^2 + (50)^2	(6)(49) + (7)(50)	(42) (52)	(14)(49) + (13)(50)	(28)(54)
1	-2.19977	-.77632	5.44167	-21.43194	-15.25526	.77256	.08710
2	-2.93029	-.93926	9.46878	-23.00581	-16.56188	1.20634	.20504
3	-3.33546	-.84631	11.84153	-16.09609	-11.82741	1.36395	.30604
4	-3.87870	-.19736	15.08327	-15.94513	-11.40236	1.52860	.41418
5	-3.85006	.42873	15.00678	-13.25996	-9.68375	1.39827	.45014
6	-3.85992	.89418	15.69854	-10.33445	-7.45424	1.32863	.48413
7	-3.67745	1.28538	15.17586	-7.79158	-5.75798	1.25158	.50860
8	-3.49294	1.54621	14.59140	-5.89752	-4.22498	1.08491	.48405
9	-3.01197	1.98123	12.99724	-4.18303	-3.08519	.92071	.46873
10	-2.54194	2.05856	10.69914	-2.32136	-1.70736	.63986	.36590
11	-2.16033	2.12266	9.17271	-2.34895	-1.68396	.53898	.32722
12	-1.77580	2.10523	7.58548	-1.21118	-.88840	.41977	.30229
13	-1.43630	1.98053	5.98547	-.71219	-.51755	.27959	.22226
14	-1.16985	1.85177	4.79759	-.53952	-.39585	.21426	.19660
15	-.92278	1.73904	3.87580	-.24447	-.17756	.16647	.16850
16	-.77976	1.56208	3.04812	-.07654	-.-5618	.12976	.14940
17	-.54182	1.47956	2.48268	-.26246	-.19049	.10640	.13405
18	-.50000	1.23596	1.77759	-.08572	-.06414	.08219	.12438
19	-.40750	1.20942	1.62875	-.17934	-.13013	.06755	.10420
20	-.36412	1.09086	1.32256	-.12739	-.09288	.06024	.10233
21	-.29077	1.06279	1.21406	-.07516	-.05596	.05288	.09832
22	-.23029	.93371	.92485	-.13288	-.09892	.04332	.08786
	-43.35784		169.81986		-91.31242		5.79131

TABLE 2
METHOD OF SOLVING NORMALIZED LEAST-SQUARES EQUATIONS WITH FOUR UNKNOWNS

Given Equations:

$$a_{11}W + a_{12}X + a_{13}Y + a_{14}Z = b_1$$
$$a_{21}W + a_{22}X + a_{23}Y + a_{24}Z = b_2$$
$$a_{31}W + a_{32}X + a_{33}Y + a_{34}Z = b_3$$
$$a_{41}W + a_{42}X + a_{43}Y + a_{44}Z = b_4$$

Augmented Matrix:

$$\begin{matrix} a_{11} & a_{12} & a_{13} & a_{14} & b_1 \\ a_{21} & a_{22} & a_{23} & a_{24} & b_2 \\ a_{31} & a_{32} & a_{33} & a_{34} & b_3 \\ a_{41} & a_{42} & a_{43} & a_{44} & b_4 \end{matrix}$$

Auxiliary Matrix:

$A_{11} =$ a_{11}	$A_{12} =$ a_{12}/A_{11}	$A_{13} =$ a_{13}/A_{11}	$A_{14} =$ a_{14}/A_{11}	$B_1 =$ b_1/A_{11}
$A_{21} =$ a_{21}	$A_{22} =$ $a_{22}-A_{12}A_{21}$	$A_{23} =$ $(a_{23}-A_{13}A_{21})/A_{22}$	$A_{24} =$ $(a_{24}-A_{14}A_{21})/A_{22}$	$B_2 =$ $(b_2-B_1A_{21})/A_{22}$
$A_{31} =$ a_{31}	$A_{32} =$ $a_{32}-A_{12}A_{31}$	$A_{33} =$ $a_{33}-A_{13}A_{31}-A_{23}A_{32}$	$A_{34} =$ $(a_{34}-A_{14}A_{31}-A_{24}A_{32})/A_{33}$	$B_3 =$ $(b_3-B_1A_{31}-B_2A_{32})/A_{33}$
$A_{41} =$ a_{41}	$A_{42} =$ $a_{42}-A_{12}A_{41}$	$A_{43} =$ $a_{43}-A_{13}A_{41}-A_{23}A_{42}$	$A_{44} =$ $a_{44}-A_{14}A_{41}-A_{24}A_{42}-A_{34}A_{43}$	$B_4 =$ $(b_4-B_1A_{41}-B_2A_{42}-B_3A_{43})/A_{44}$

Evaluation of Unknowns from Auxiliary Matrix:

$$Z = B_4 \qquad X = B_2 - ZA_{24} - YA_{23}$$
$$Y = B_3 - ZA_{34} \qquad W = B_1 - ZA_{14} - XA_{12}$$

TABLE 3

SAMPLE SOLUTION OF NORMALIZED EQUATIONS FOR STABILITY DERIVATIVES

Columns Summed in Table 1 to Obtain Augmented Matrix:

22	21	39	-43	-47
21	m	49	-45	-48
39	49	51	-53	55

(See Theory, Equation 19)

Augmented Matrix:

20.72385	-2.10654	17.89218	107.27281	-13.30576
-2.18654	22	-43.35784	-4.65182	-18.33820
17.89218	-43.35784	169.81986	91.31242	5.79131

Auxiliary Matrix:

20.72385	-.10165	.86337	5.17634	-.64206
-2.10654	21.78587	-1.90670	.28699	-.90383
17.89218	-41.53912	75.16967	.14125	-.26959

Solutions from Auxiliary Matrix

$C_{L_{\dot{\theta}}} = .14125$

$C_{L_{\delta}} = .28699 - .14125 \times (-1.90670) = .55631$

$C_{L_{\alpha}} = 5.17634 - .14125 \times .86337 - .55631 \times (-.10165) = 5.11094$

$C_{m_{\dot{\theta}}} = -.26959$

$C_{m_{\delta}} = -.90383 - (-.26959) \times (-1.90670) = -1.41786$

$C_{m_{\alpha}} = -.64206 - (-.26959) \times .86337 - (-1.41786) \times (-.10165) = -.55342$

If $C_{L_{\dot{\theta}}}$ is assumed to = 0, then:

$C_{L_{\delta}} = .28699$

$C_{L_{\alpha}} = 5.17634 - .28699 \times (-.10165) = 5.20551$

If $C_{L_{\dot{\theta}}}$ and $C_{L_{\delta}}$ are assumed to = 0, then:

$C_{L_{\alpha}} = 5.17634$

TABLE 4

SAMPLE SOLUTION FOR THE PITCHING VELOCITY TRANSFER FUNCTION CONSTANTS

Columns Summed in Table 1 to Obtain Augmented Matrix:

m	0	25	-13	-30
0	23	-30	-25	-25
25	-30	29	0	0
-13	-25	0	12	29

(Reference Theory, Equation 24)

Augmented Matrix:

22	0	100.54611	30.15384	295.41151
0	679.0712	295.41151	-100.54611	-4690.1187
100.54611	295.41151	960.15451	0	0
30.15384	-100.54611	0	88.81077	960.15451

Auxiliary Matrix:

22	0	4.57027	1.37062	13.42779
0	679.0712	.43502	-.14806	-6.90666
100.54611	295.41151	372.12172	-.25279	1.85475
30.15384	-100.54611	-94.07162	8.81409	4.00456

Solutions from Auxiliary Matrix:

$$\bar{A}_0 = 4.00456$$

$$\bar{A}_1 = 1.85475 - 4.00456 \times (-.25279) = 2.86706$$

$$\bar{B}_1 = -6.90666 - 4.00456 \times (-.14806) - 2.86706 \times .43502 = -7.56097$$

$$\bar{B}_0 = 13.42779 - 4.00456 \times 1.37062 - 2.86706 \times 4.57027 = -5.16418$$

TABLE 5

SAMPLE SOLUTION FOR ANGLE OF ATTACK TRANSFER FUNCTION CONSTANTS

Columns Summed in Table 1 to Obtain Augmented Matrix:

m	0	-17	-21	-31
0	23	-31	17	-36
-17	-31	32	0	0
-21	17	0	22	32

(See Theory, Equation 25)

Augmented Matrix:

22	0	29.34224	2.10654	-72.87690
0	679.0712	-72.87690	-29.34224	-411.54432
29.34224	-72.87690	73.66311	0	0
2.10654	-29.34224	0	20.72385	73.66311

Auxiliary Matrix:

22	0	1.33373	.09575	-3.31258
0	679.0712	-.10731	-.04320	-.60604
29.34224	-72.87690	26.70806	-.22307	1.98562
2.10654	-29.34224	-5.95827	17.92545	4.16667

Solutions from Auxiliary Matrix:

$$\bar{A}_0 = 4.16667$$

$$\bar{A}_1 = 1.98562 - (-.22307) \times 4.16667 = 2.91508$$

$$\bar{C}_1 = -.60604 - (-.04320) \times 4.16667 - (-.10731) \times 2.91508 = -.11322$$

$$\bar{C}_0 = -3.31258 - .09575 \times 4.16667 - 1.33373 \times 2.91508 = -7.59947$$

TABLE 6

SAMPLE SOLUTION FOR NORMAL ACCELERATION TRANSFER FUNCTION CONSTANTS

Columns Summed in Table 1 to Obtain Augmented Matrix:

m	0	-6	26	-23	-33
0	23	-26	-33	0	-37
-6	-26	5	0	33	35
26	-33	0	34	-37	0
-23	0	33	-37	24	39

(Reference Theory, Equation 26)

Augmented Matrix:

22	0	-6.91452	-227.21809	-679.0712	955.47050
0	679.0712	227.21809	955.47050	0	4195.8905
-6.91452	227.21809	1070.17454	0	-955.47050	4941.9625
-227.21809	955.47050	0	4941.9625	4195.8905	0
-679.0712	0	-955.47050	4195.8905	48370.050	-63366.258

Auxiliary Matrix:

22	0	-.31429	-10.32809	-30.86687	43.43047
0	679.0712	.33460	1.40702	0	6.17886
-6.91452	227.21809	991.97420	-.39427	-1.17835	3.86936
-227.21809	955.47050	-391.11280	1096.66347	-2.98951	4.99499
-697.0712	0	-1168.89579	-3278.47851	16230.8349	-.79940

Solutions from Auxiliary Matrix:

$\bar{E}_2 = -.79940$

$\bar{A}_1 = 2.60518$

$\bar{A}_0 = 3.95453$

$\bar{E}_1 = 1.19013$

$\bar{E}_0 = 46.90490$

TABLE 7

TRANSFER FUNCTION CONSTANTS COMPUTED FROM FLIGHT DATA

Parameters Obtained From Flight Test Data:

$h_{ave} = .180\ \text{sec}^2$	$C_{m_\alpha} = -.553$	$C_{L_\alpha} = 5.111$
$\tau_{ave} = 3.00\ \text{sec}$	$C_{m_\delta} = -1.418$	$C_{L_\delta} = .556$
$V_{ave} = 265\ \text{ft/sec}$	$C_{m_{\dot\theta}} = -.270\ \text{sec}$	$C_{L_{\dot\theta}} = .140\ \text{sec}$
$g = 32.2\ \text{ft/sec}^2$	$C_{m_{\dot\alpha}} = -.122\ \text{sec}$	$C_{L_{\dot\alpha}} = .063\ \text{sec}$

Transfer Function Constants Calculated from Flight Test Parameters:

$$A_2 = -hC_{L_{\dot\alpha}} - 2h\tau = -1.091\ \text{sec}^3$$

$$\bar{A}_0 = (C_{L_\alpha} C_{m_{\dot\theta}} - C_{m_\alpha} C_{L_{\dot\theta}} + 2\tau C_{m_\alpha})/A_2 = 4.254\ \text{sec}^{-2}$$

$$\bar{A}_1 = (C_{L_{\dot\alpha}} C_{m_{\dot\theta}} + 2\tau C_{m_{\dot\theta}} - hC_{L_\alpha} - C_{m_{\dot\alpha}} C_{L_{\dot\theta}} + 2\tau C_{m_{\dot\alpha}}/A_2 = 2.998\ \text{sec}^{-1}$$

$$\bar{B}_0 = (C_{L_\delta} C_{m_\alpha} - C_{L_\alpha} C_{m_\delta})/A_2 = -6.359\ \text{sec}^{-3}$$

$$\bar{B}_1 = (C_{m_{\dot\alpha}} C_{L_\delta} - C_{L_{\dot\alpha}} C_{m_\delta} - 2\tau C_{m_\delta})/A_2 = -7.816\ \text{sec}^{-2}$$

$$\bar{C}_0 = (C_{m_\delta} C_{L_{\dot\theta}} - C_{L_\delta} C_{m_{\dot\theta}} - 2\tau C_{m_\delta})/A_2 = -7.752\ \text{sec}^{-2}$$

$$\bar{C}_1 = hC_{L_\delta}/A_2 = -0.092\ \text{sec}^{-1}$$

$$\bar{E}_0 = V(C_{m_\delta} C_{L_\alpha} - C_{L_\delta} C_{m_\alpha})/gA_2 = 52.334\ \text{sec}^{-2}$$

$$\bar{E}_1 = V(C_{m_\delta} C_{L_{\dot\theta}} - C_{L_\delta} C_{m_{\dot\theta}} + C_{L_\delta} C_{m_{\dot\alpha}} + C_{L_{\dot\alpha}} C_{m_\delta})/gA_2 = 0.527\ \text{sec}^{-1}$$

$$\bar{E}_2 = VhC_{L_\delta}/gA_2 = -0.755$$

TABLE 8

TRANSFER FUNCTION CONSTANTS FROM FLIGHT TEST DATA

$$\dot{\theta}/\delta = (\bar{B}_0 + \bar{B}_1 D)/(\bar{A}_0 + \bar{A}_1 D + D^2)$$

	From Data Points 1 through 22	From Data Points 1 through 17	From Stability Derivatives, Table 7
$\bar{A}_0$	4.005	3.840	4.234
$\bar{A}_1$	2.867	2.917	2.998
$\bar{B}_0$	-5.164	-5.116	-6.359
$\bar{B}_1$	-7.561	-7.778	-7.816

$$\alpha/\delta = (\bar{C}_0 + \bar{C}_1 D)/(\bar{A}_0 + \bar{A}_1 D + D^2)$$

$\bar{A}_0$	4.167	4.174	
$\bar{A}_1$	2.915	2.959	
$\bar{C}_0$	-7.599	-7.732	-7.752
$\bar{C}_1$	-0.113	-0.122	-0.092

$$n/\delta = (\bar{E}_0 + \bar{E}_1 D + \bar{E}_2 D^2)/(\bar{A}_0 + \bar{A}_1 D + D^2)$$

$\bar{A}_0$	3.954	4.188	
$\bar{A}_1$	2.605	2.889	
$\bar{E}_0$	46.905	51.211	52.334
$\bar{E}_1$	1.190	0.485	0.527
$\bar{E}_2$	-0.799	-0.655	-0.755

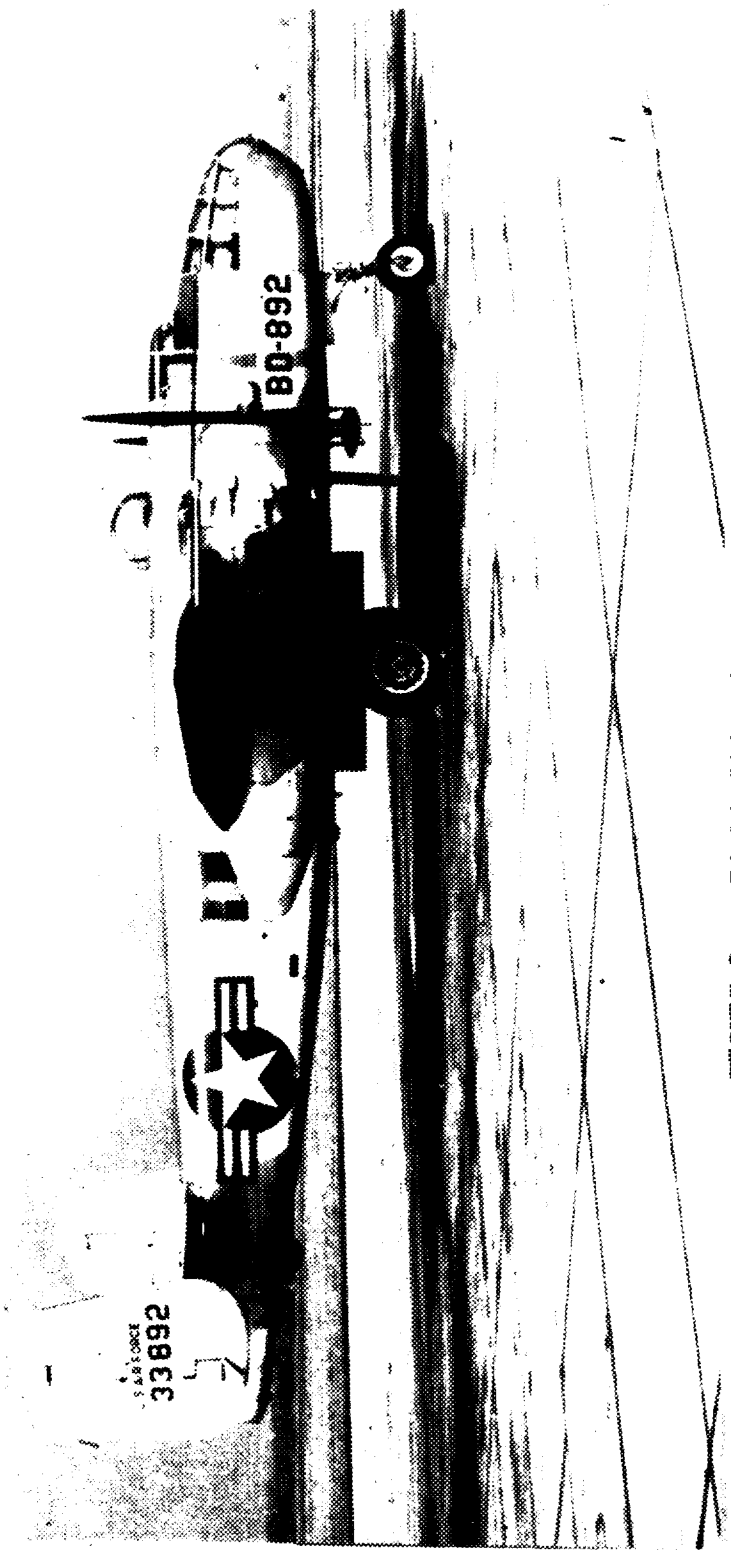

FIGURE 1. Right Side View
EB-25J 433892

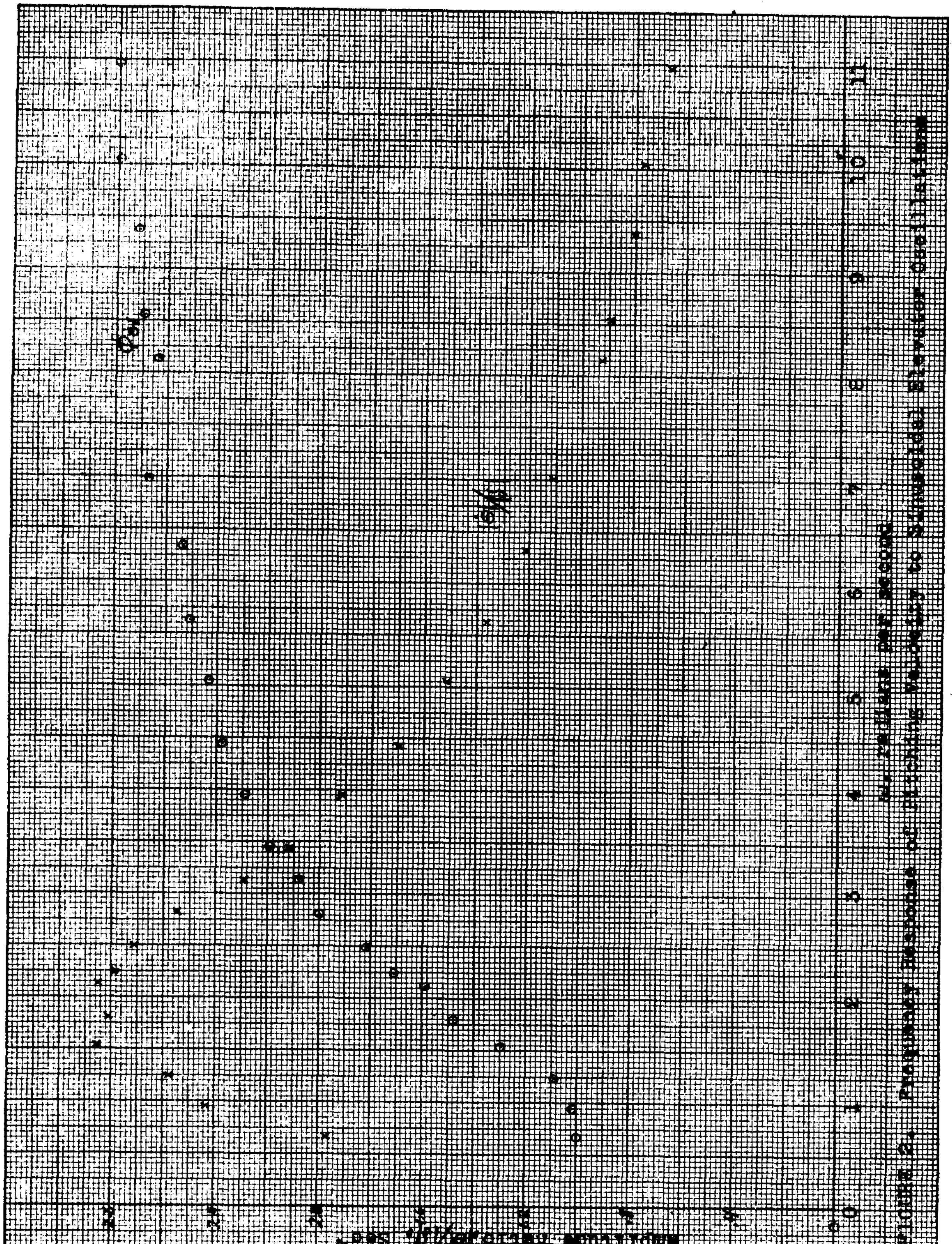

FIGURE 2. Frequency Response of Pitching Velocity to Sinusoidal Elevator Oscillations

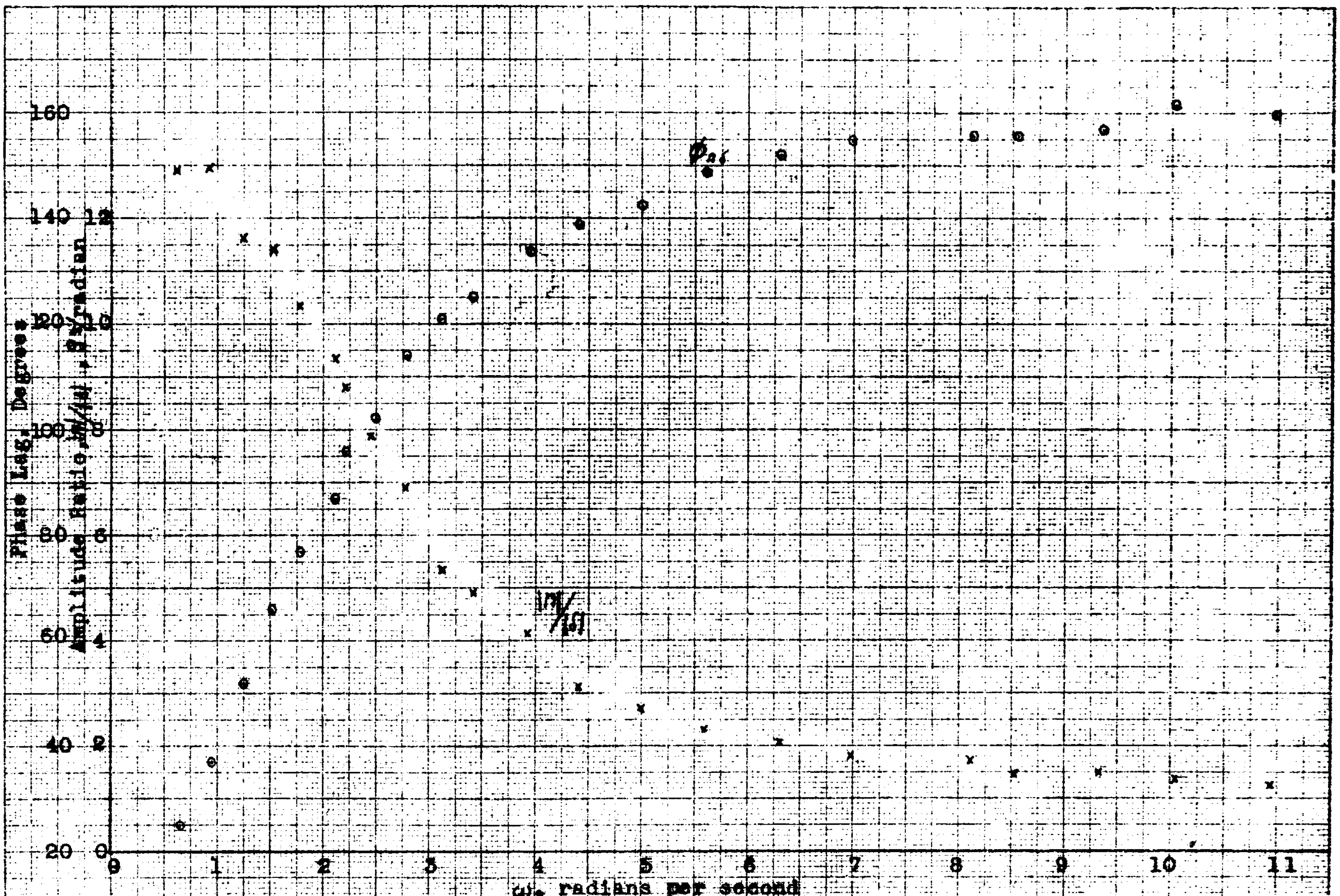

FIGURE 3. Frequency Response of Normal Acceleration to Sinusoidal Elevator Oscillations

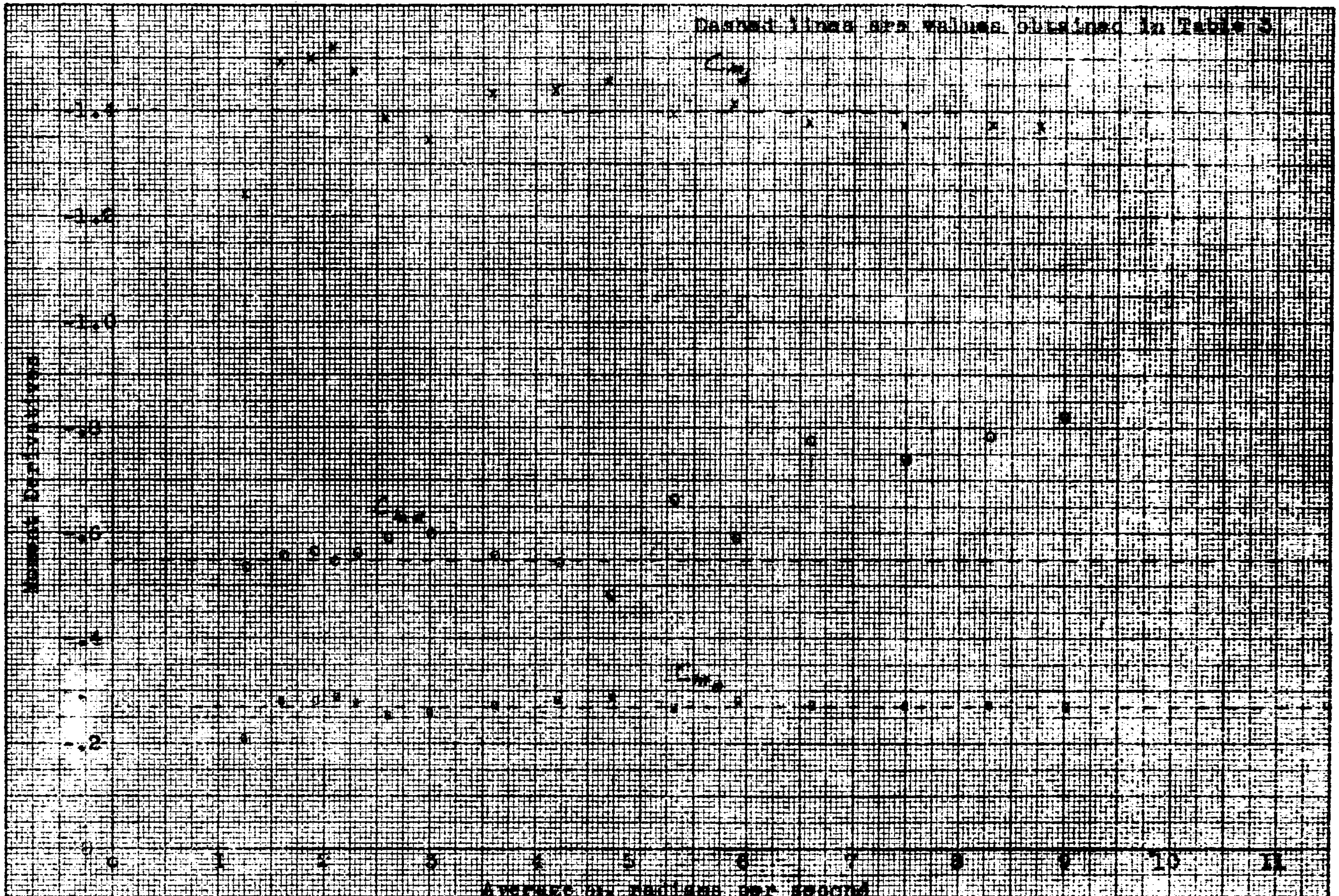

FIGURE 4. Effect of Frequency on Moment Derivatives Obtained from Flight Test Data

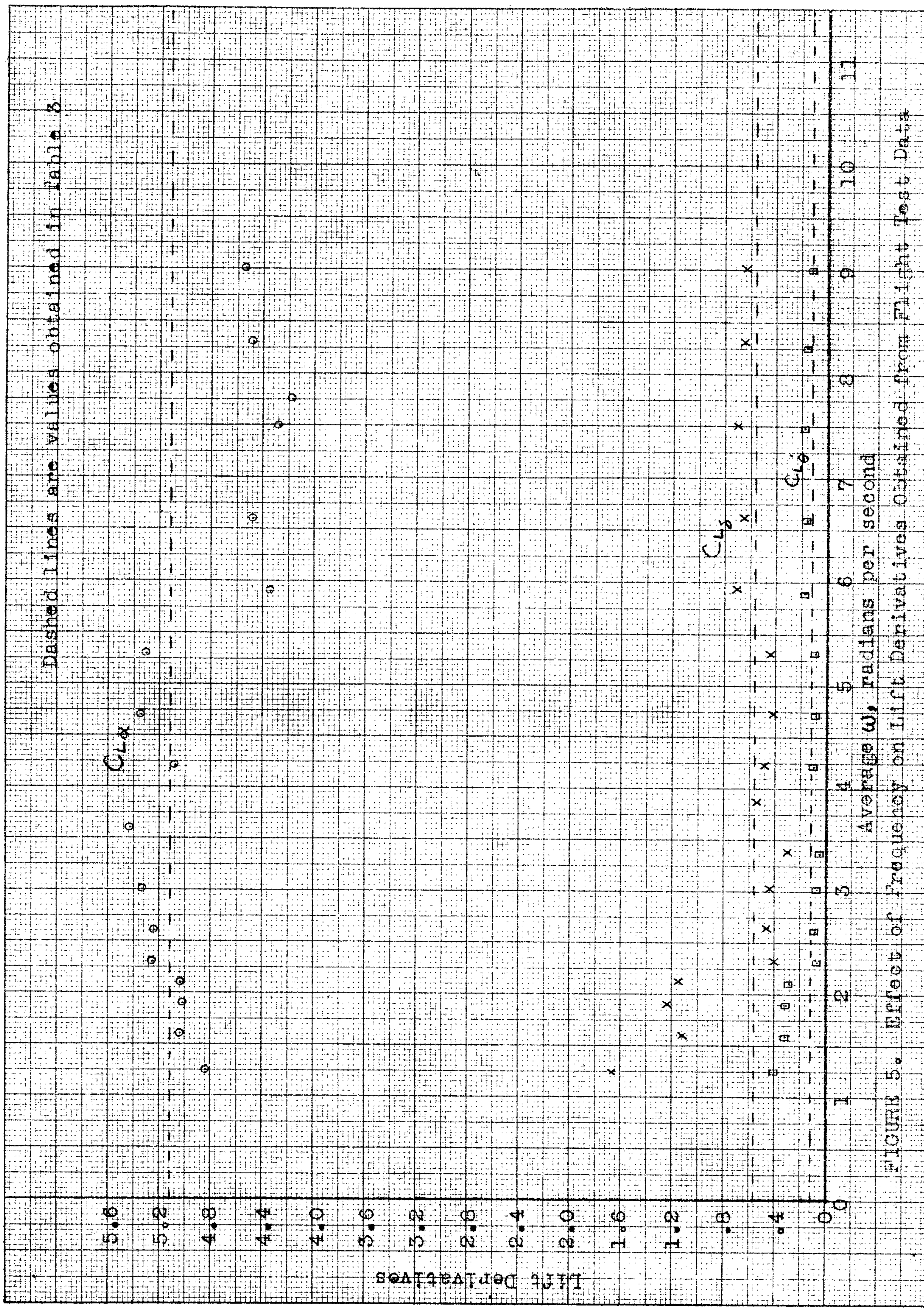

FIGURE 5. Effect of Frequency on Lift Derivatives Obtained from Flight Test Data

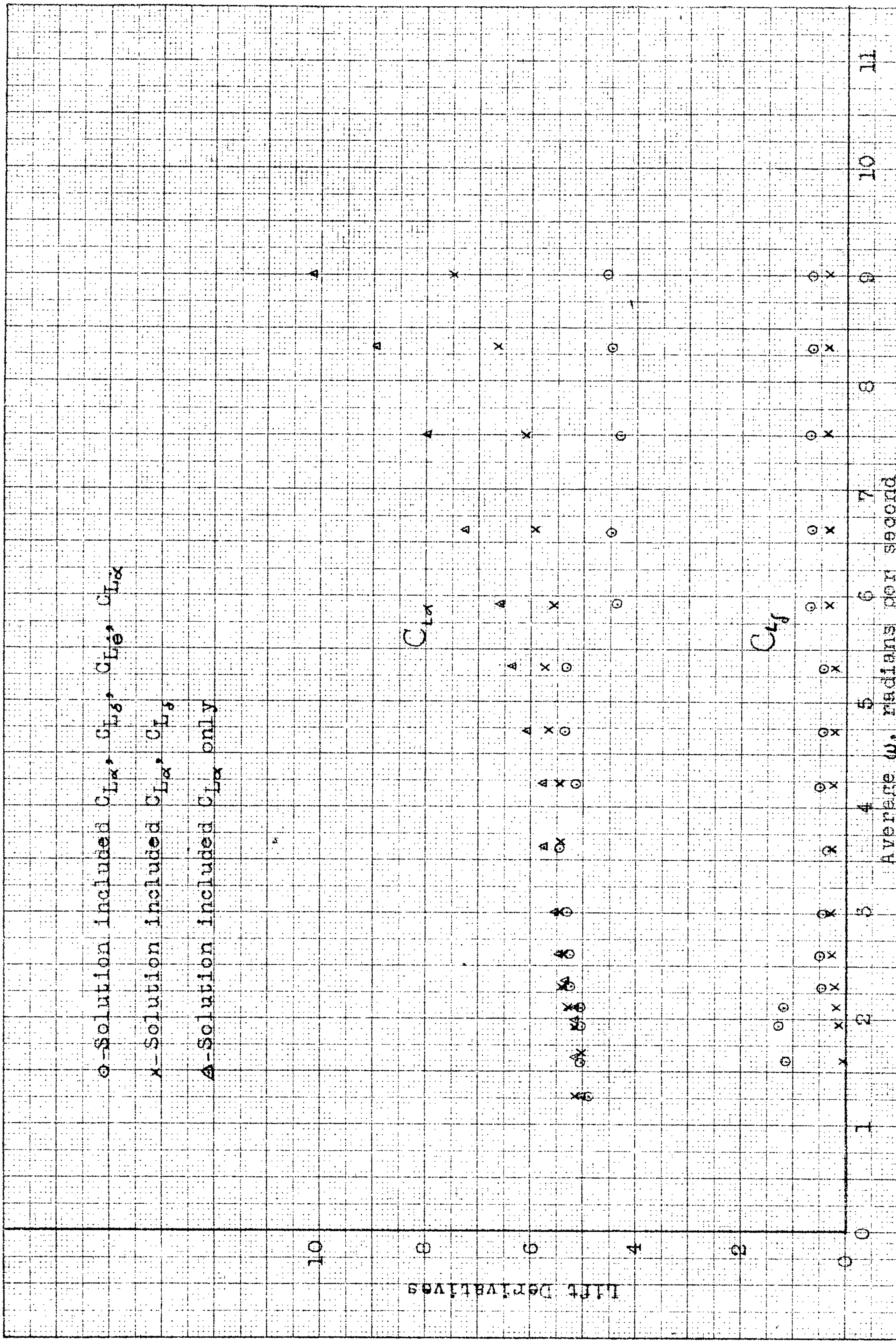

FIGURE 6. Effect of Neglecting C_{L_θ}, $C_{L_{\dot\alpha}}$, and C_{L_δ} on the Values obtained for C_{L_α} and C_{L_δ}

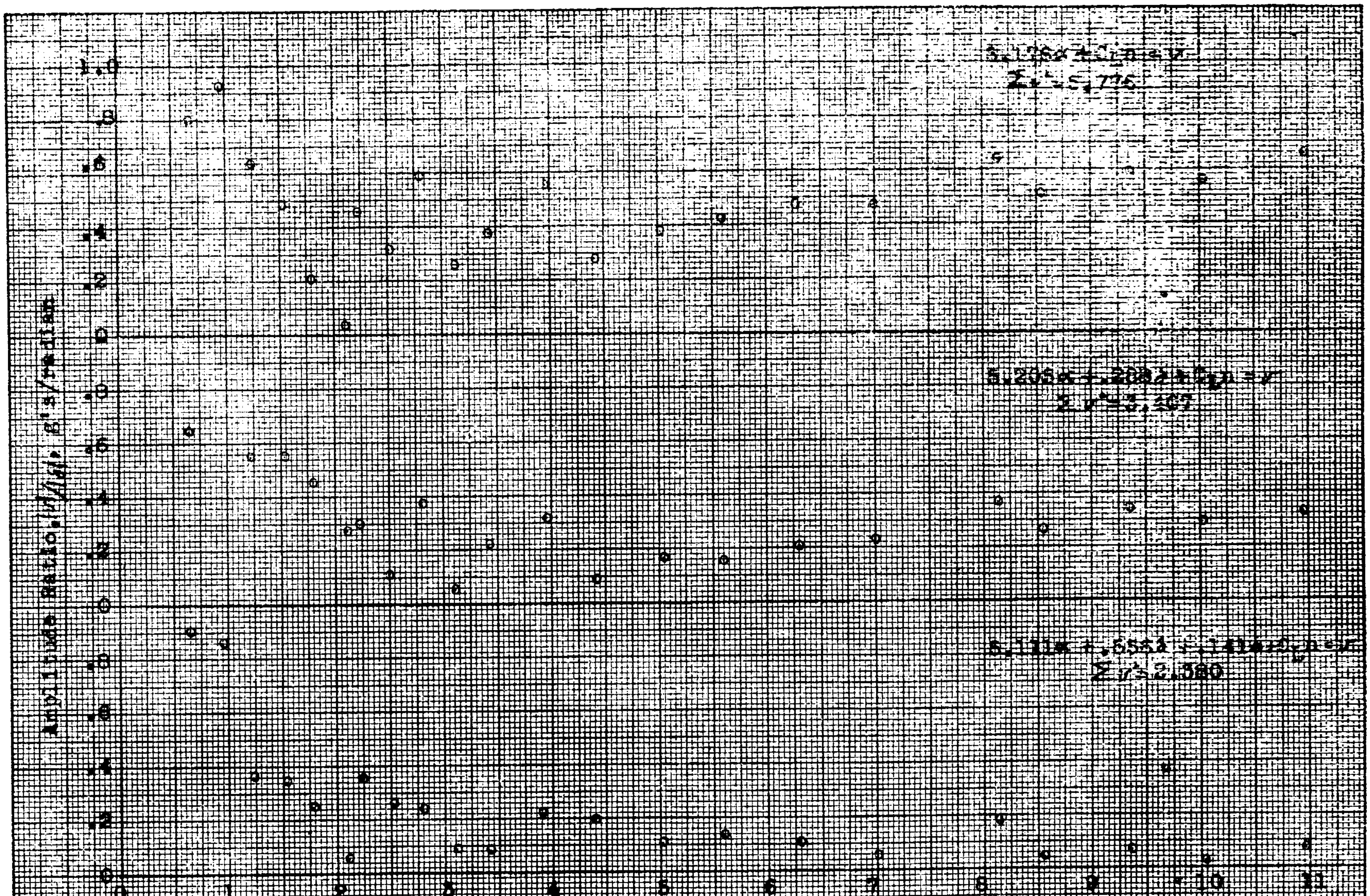

Figure 7a. Effect of Neglecting [illegible], [illegible], and [illegible] on the Residual Amplitudes

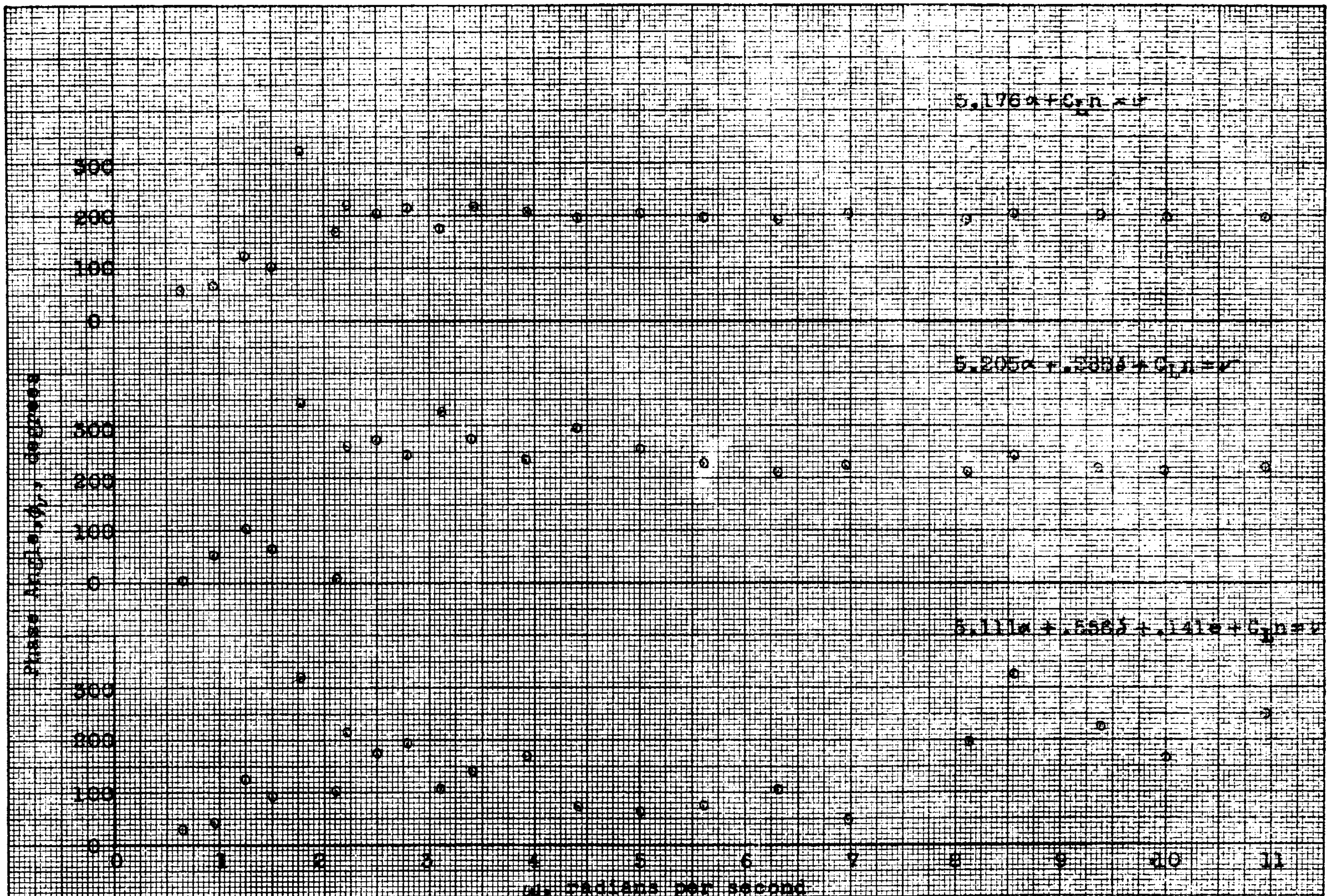

FIGURE 7b. Effect of Neglecting $C_{L_{\dot{\theta}}}$, $C_{L_{\dot{\alpha}}}$, and $C_{L_{\delta}}$ on the Residual Phase Angles

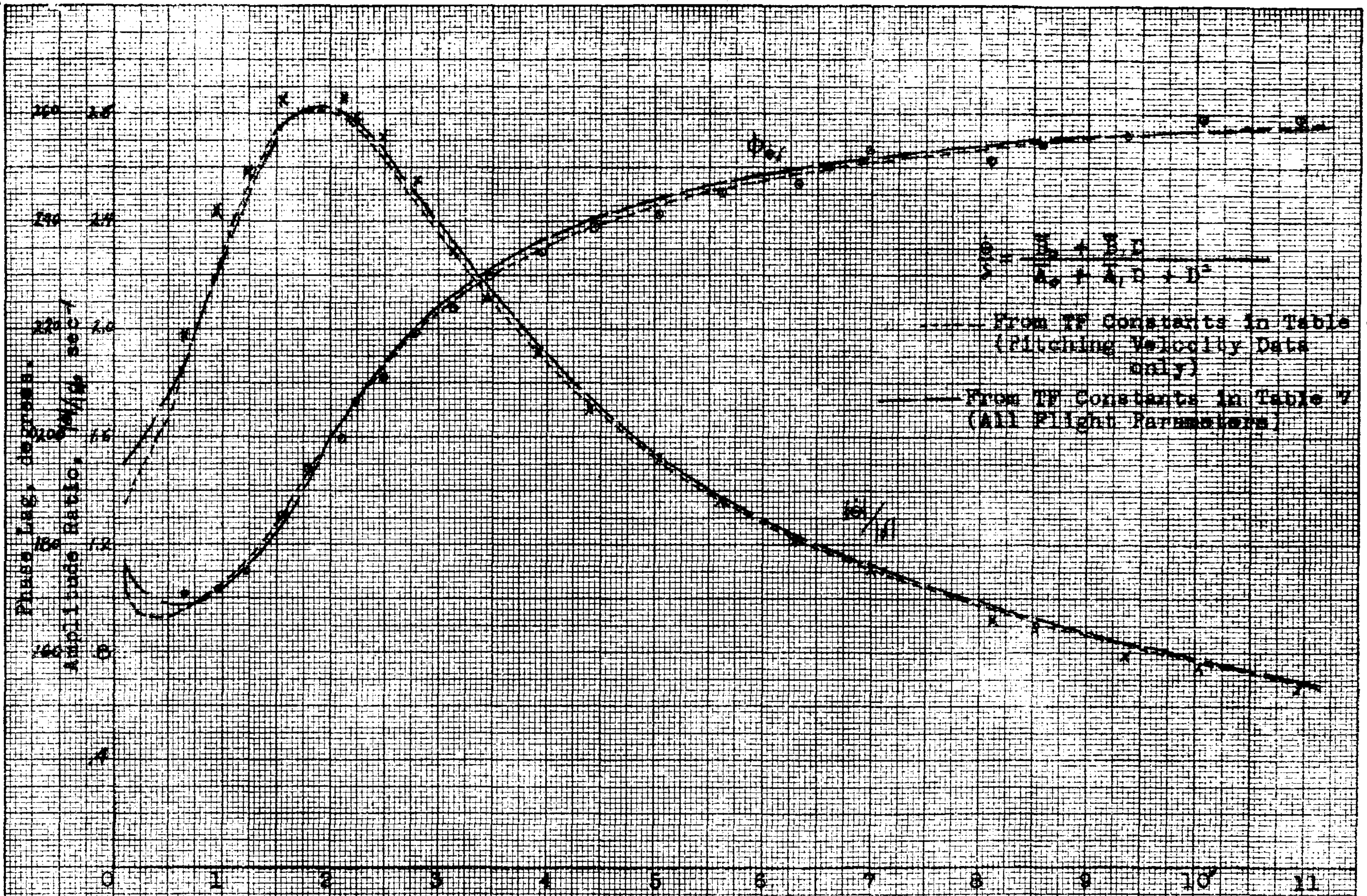

FIGURE 8. Pitching Velocity Frequency Response Computed from Transfer Function Constants and Compared with Flight Data.

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